AN INVITATION TO COMPLEX NON-KÄHLER GEOMETRY

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Notes for a course given at FaMAF, Universidad Nacional de Córdoba, Argentina; and at the 14th International Young Researcher Workshop on Geometry, Mechanics and Control, Georg-August-Universität Göttingen.

ABSTRACT. Kähler geometry provides analytic techniques to extend results from the algebraic to the transcendental setting. Non-Kähler geometry is then the attempt to perform a separate analysis of the complex and symplectic contributions. Cohomological invariants and canonical Hermitian metrics are useful tools for a tentative classification of compact complex manifolds.

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Introduction

"*It's not mathematics that you need to contribute to. It's deeper than that: how might you contribute to humanity, and even deeper, to the well-being of the world, by pursuing mathematics?*" (Bill Thurston)

We give an introduction on complex manifolds, focused on cohomologies and Hermitian metrics.

We start by introducing complex manifolds as the objects in the category of holomorphic maps. We study the double complex of forms, its decomposition following Khovanov and Stelzig, and the cohomologies associated to it. Beside de Rham (*i.e.* total) and Dolbeault (*i.e.* column) cohomology, we can define also Bott-Chern and Aeppli cohomologies as "bridges" to compare holomorphic and topological information. We give sheaf-theoretic and analytic interpretations of these cohomologies; we also discuss their symmetries; we deduce some inequalities between Betti numbers and Hodge and Bott-Chern numbers; we characterize isomorphisms between cohomologies; and, finally, we consider locally homogeneous manifolds of nilpotent Lie groups as a class of interesting examples for explicit computations.

In the second lecture, we focus on the property when the decomposition of forms move in cohomology, namely, the so-called "$\partial\overline{\partial}$-Lemma property". It is a foundational result in Kähler geometry, yielding Hodge decomposition in cohomology. In particular, we study its behaviour under deformations of the complex structure and under modifications, to the attempt to take advantage of this property in bimeromorphic classification of compact complex manifolds. We sketch the parallel of the theory for symplectic structures, that frames in the more general context of generalized-complex geometry. We also briefly discuss the algebraic structure of cohomology in view of formality, that represents a topological obstruction to the validity of $\partial\overline{\partial}$-Lemma, with the further scope to understand better the role for Bott-Chern cohomology.

Among special metrics, Kähler metrics play a special role, and the tentative to generalize their properties to a wider context is one of the main aims in complex non-Kähler geometry. We already briefly resumed the analytic and transcendental techniques that allow to get Hodge decomposition and further results on compact Kähler manifolds. (By the way, we just cite locally conformally geometry as a first generalization of the Kähler condition, in other works, as a "equivariant (homothetic) Kähler geometry". This is a class in the classification of Gray-Hervella [GH80].) We then investigate the existence of canonical metrics in Hermitian geometry: specifically, as a case study, we propose an analogue of the Yamabe problem in the Hermitian setting, seeking for constant scalar curvature metrics with respect to the Chern connection in a conformal class.

Acknowledgement. These draft "notes" have to be intended just as "notes in progress", and as such there are plenty of mistakes, inaccuracies, they are poorly organized, many subjects and references are missing: we are sorry about that. Any suggestion and correction is warmly welcome!

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Lecture I. Cohomological invariants of complex manifolds

Complex manifolds are the objects in order to define the category of holomorphic maps. In this section, we first recall some properties of *holomorphic functions* (see *e.g.* the references [dSSST06, GR09, Hör90, Car95, Nar95, LT11, Gun90a, Gun90b, Gun90c] for a more complete and detailed account). Then we recall some basic properties of *(almost-)complex manifolds* (references for this chapter are *e.g.* [Huy05, Dem12, Kod05, MK06, GH94, Voi02a, Bal06, Mor07, BHPVdV04, Gau15, Uen75] and many others). We then focus on the differential analysis on complex manifolds, and we introduce the *double complex of forms* and their *cohomologies*. On compact complex manifolds, we have an *elliptic Hodge theoretic* interpretation. Finally, we introduce *nilmanifolds*, namely, compact locally-homogeneous manifolds of connected simply-connected nilpotent Lie group, endowed with suitable invariant complex structures, as a class of example where cohomology can be explicity computed by reducing to the Lie algebra structure.

The main references for this Lecture are [Huy05, Ste18a, Sch07, Rol11, Ang14].

I.1. Complex and almost-complex manifolds

I.1.1. **The algebra of holomorphic functions.** We briefly recall that a function $f \colon \Omega \to \mathbb{C}$ of class $\mathcal{C}^1$ defined on a domain $\Omega \subseteq \mathbb{C}^n$ is *holomorphic* if, at any point $x \in \Omega$, its differential $df\lfloor_x \colon T_x\Omega \simeq \mathbb{C}^n \to \mathbb{C}$ is $\mathbb{C}$-linear. In fact, by the Osgood theorem [Osg32], f is holomorphic if and only if it is continuous and it is separately holomorphic in each variable, that is, it satisfies the *Cauchy-Riemann equations*

$$\text{for any } k, \qquad \frac{\partial}{\partial \bar{z}^k} f = 0. \tag{CR}$$

Here, we are denoting coordinates on $\mathbb{C}^n$, as complex space, by $(z^j = x^j + \sqrt{-1}y^j)_j$. The differential of $f \in \mathcal{C}^1(\Omega; \mathbb{C})$ splits as

$$df = \frac{\partial f}{\partial z^j} dz^j + \frac{\partial f}{\partial \bar{z}^j} d\bar{z}^j,$$

where the Einstein notation is assumed; more precisely, one computes

$$\frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - \sqrt{-1}\frac{\partial}{\partial y^j}\right), \qquad \frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + \sqrt{-1}\frac{\partial}{\partial y^j}\right)$$

in terms of standard real coordinates $(x^j, y^j)_j$. In particular, it follows that real and imaginary parts of a holomorphic functions $f = u + \sqrt{-1}\,v$ are *pluriharmonic*, that is, harmonic when restricted to any complex line.

The homogenous Cauchy-Riemann system (CR) is *elliptic*, see *e.g.* [Wel08, Example 2.6]. This has several consequences.

- The Green function identity for Dirichlet boundary value problem gives the *Cauchy integral formula*:
  for any $z \in P(z_0; R) \Subset \Omega$, where $z_0 \in \Omega$ and $R \in \mathbb{R}^n$ are fixed, we have the integral representation
  $$f(z) = (2\pi\sqrt{-1})^{-n} \int_{bP(z_0;R)} f(\zeta_1, \dots, \zeta_n) \frac{d\zeta_1 \wedge \cdots \wedge d\zeta_n}{\prod_j (\zeta_j - z_j)}.$$
  (Here, *polydiscs* $P(z_0; R) := \prod_{j=1}^n D(z_0^j; R_j)$ and their *Silov boundaries* $bP(z_0; R) := \prod_{j=1}^n bD(z_0^j; R_j)$ take the place of discs and their boundaries in view of the separately holomorphicity.)
- Hypoellipticity yields *elliptic regularity*:
  indeed, as a consequence of Cauchy integral formula, the *Cauchy estimates*
  $$\left| \frac{\partial^{m_1+\cdots+m_n} f}{(\partial z_1)^{m_1} \cdots (\partial z_n)^{m_n}}(z_0) \right| \leq \prod_j \frac{m_j!}{R_j^{m_j}} \cdot \sup_{P(z_0;R)} |f|$$
  yield that holomorphic functions are $\mathcal{C}^\infty$; moreover, holomorphic functions are also complex-analytic: for any point $z_0 \in \Omega$, we have the *power series expansion*
  $$f(z) = \sum_{m \in \mathbb{N}^n} a_m \cdot (z - z_0)^m$$
  being normally convergent (*i.e.* uniformly convergent on compact subsets) in some neighbourhood of z_0. Here, one computes
  $$a_m = \frac{1}{\prod_j m_j!} \cdot \frac{\partial^{m_1+\cdots+m_n} f}{(\partial z_1)^{m_1} \cdots (\partial z_n)^{m_n}}(z_0).$$

- *Maximum principle* assures unique continuation:
  if $|f|$ admits a local maximum at a point $z_0 \in \Omega$, then f is constant in a neighbourhood of z_0.

Later, we will use the following *local solvability result*:

Theorem I.1.1 (Dolbeault-Grothendieck Lemma). *Let α be a form on $\mathbb{C}^n$ of type (p,q), for $q \geq 1$, and in class $\mathcal{C}^k$, for $k \geq 1$. That is, $\alpha = \sum_{I,J} \alpha_{I,\bar{J}} dz^I \wedge d\bar{z}^J$, where I and J are multi-indices of length p and q respectively, and $\alpha_{I,\bar{J}} \in \mathcal{C}^k(\mathbb{C}^n)$. If $\overline{\partial}\alpha = 0$, (that is, $\sum_k \sum_{I,J} \frac{\partial \alpha_{I,\bar{J}}}{\partial \bar{z}^k} d\bar{z}^k \wedge dz^I \wedge d\bar{z}^J = 0$,) then there exists a form β of type $(p, q-1)$ and in class $\mathcal{C}^k$ such that $\overline{\partial}\beta = \alpha$.*

For vector-valued functions $f = (f^1, \ldots, f^m)\colon D \to D'$ for domains $D \subseteq \mathbb{C}^n$ and $D' \subseteq \mathbb{C}^m$, *holomorphicity* is meant for any component f^j. The *complex Jacobian matrix* is

$$J^{\mathbb{C}}(f) := \left(\frac{\partial f^\alpha}{\partial z^\beta} \right)_{\alpha,\beta}.$$

From the real point of view: the real Jacobian determinant satisfies

$$\det J^{\mathbb{R}}(f) = |\det J^{\mathbb{C}}(f)|^2. \tag{1}$$

We recall two basic results:

- *inverse function theorem*:
  for $n = m$, if $\det J^{\mathbb{C}}(f)(z_0) \neq 0$, then there exists an open neighbourhood U of z_0 such that $f\lfloor_U \colon U \to f(U)$ is biholomorphic;
- *implicit function theorem*:
  for $m \leq n$, if $\operatorname{rk} J^{\mathbb{C}}(f)(z_0) = m$, then there exist open subsets $D_1 \subseteq \mathbb{C}^m$ and $D_2 \subseteq \mathbb{C}^{n-m}$ such that $z_0 \in D_1 \times D_2 \subseteq D$, and a holomorphic map $g\colon D_1 \to D_2$, such that
  $$f(z) - f(z_0) = 0 \qquad \text{if and only if}$$
  $$z = (z_1, \ldots, z_m, g_1(z_1, \ldots, z_m), \ldots, g_{n-m}(z_1, \ldots, z_m)).$$
  It gives local representation of complex submanifolds as zero locus of analytic functions.

I.1.2. **Complex manifolds.** We recall that a *holomorphic manifold* (usually called as *complex manifold*) is a second-countable Hausdorff topological space X with a covering $X = \bigcup_j U_j$ where U_j are open sets homeomorphic to open set in $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ via $\phi_j\colon U_j \to \mathbb{C}^n$, such that the transition functions $\phi_k^{-1} \circ \phi_j\colon \phi_j(U_j \cap U_k) \to \phi_k(U_j \cap U_k)$ are holomorphic. (In particular, X is a paracompact topological space and a differentiable manifold of dimension $2n$.) Note that n is actually independent of U_j, and it is the *complex dimension* of X. The definition allows to define the notion of *holomorphic map* $f\colon X \to Y$ between complex manifolds, by requiring that f is holomorphic when read in local charts.

In sheaf-theoretic terms, we can interpret complex manifolds as reduced local ringed spaces over $\mathbb{C}$ with structure sheaf locally isomorphic to the model space $(\Omega, \mathcal{O}_\Omega)$, where $\Omega \subset \mathbb{C}^n$ open subset, $\mathcal{O}_\Omega$ sheaf of holomorphic functions over Ω, see *e.g.* [Dem12].

A *holomorphic vector bundle* of rank m over the complex manifold X is given by a complex manifold E, and a holomorphic map $\pi\colon E \to X$ that is *locallly trivial*, namely, there exists a covering $\{U_j\}_j$ of X and trivializations $\tau_j\colon \pi^{-1}(U_j) \stackrel{\simeq}{\to} U_j \times \mathbb{C}^m$ such that they respect the first projection $\mathrm{pr}_1\colon U_j \times \mathbb{C}^m \to U_j$, *i.e.* $\mathrm{pr}_1 \circ \tau_j = \pi$, and the transitions functions $\tau_k \circ \tau_j^{-1}$ are $\mathbb{C}$-linear on each fibre $E_x := \pi^{-1}(x)$ and holomorphic in the variable $x \in X$, so to be described by matrices with holomorphic coefficients. A *holomorphic section* of $E \stackrel{\pi}{\to} X$ is a holomorphic map $\sigma\colon X \to E$ such that $\pi \circ \sigma = \mathrm{id}$. A holomorphic vector bundle is called *trivial* if it admits a global trivialization as product. The notion of *morphism of vector bundles* is clear. The construction of tensor products, symmetric and exterior products, dual is clear. In particular, a holomorphic bundle of rank 1 is called a *line bundle.* There are some natural holomorphic vector bundles associated to a complex manifold X of complex dimension n, in particular: the *holomorphic tangent bundle* $\Theta_X \to X$, of rank n; the *holomorphic cotangent bundle* $\Omega_X \to X$ as the dual of the holomorphic tangent bundle; the *bundle of holomorphic p-forms* $\Omega^p_X := \wedge^p \Theta^*_X \to X$ as the exterior power of the holomorphic cotangent bundle; and finally the *canonical line bundle* $K_X := \det \Omega_X = \wedge^n \Omega_X$.

I.1.3. **Almost-complex manifolds.** In particular, any complex manifold of complex dimension n is a differentiable manifold of dimension $2n$. Moreover, it is orientable, thanks to (1). Furthermore, it is locally "modeled" on $\mathbb{C}^n$, whence its tangent spaces are complex vector spaces, varying smoothly with the point. In other words, the underlying differentiable manifold admits an *almost-complex structure*, that is, an endomorphism $J \in \mathrm{End}(TX)$ of the tangent bundle such that $J^2 = -\mathrm{id}$. In local holomorphic coordinates $\left(z^j =: x^{2j-1} + \sqrt{-1} x^{2j}\right)_j$, we have

$$J\left(\frac{\partial}{\partial x^{2j-1}}\right) = \frac{\partial}{\partial x^{2j}}, \qquad J\left(\frac{\partial}{\partial x^{2j}}\right) = -\frac{\partial}{\partial x^{2j-1}}$$

(note the role of Cauchy-Riemann equations in assuring that this local definition does not depend on the coordinate chart), that is:

$$J = \left(\begin{array}{cc|c|cc} & -1 & & & \\ 1 & & & & \\ \hline & & \ddots & & \\ \hline & & & & -1 \\ & & & 1 & \end{array}\right).$$

Equivalently, we can intepret an almost-complex structure as a $\mathrm{GL}(n;\mathbb{C})$-*structure*, (namely, to a $\mathrm{GL}(n;\mathbb{C})$-subbundle of the tangent frame bundle $\mathrm{GL}(X)$ of X,) and its *integrability as a* $\mathrm{GL}(n;\mathbb{C})$-*structure* corresponds to X having local holomorphic coordinates, namely, to X being a complex manifold.

Conversely, the following theorem characterizes when an almost-complex structure underlies a complex manifold:

Theorem I.1.2 (Newlander-Nirenberg integrability [NN57, Theorem 1.1])**.** *Let X be a differentiable manifold. An almost-complex structure J on X is the natural almost-complex* *structure associated to a structure of complex manifold (say,* integrable*) if and only if* $\mathrm{Nij}_J = 0$.

Here, Nij_J denotes the *Nijenhuis tensor*

$$\mathrm{Nij}_J(\_,\_) \;:=\; [\_,\_] + J\,[J\,\_,\,\_] + J\,[\_,\,J\,\_] - [J\,\_,\,J\,\_]\,,$$

that is the torsion of any connection preserving the almost-complex structure.

I.1.4. **Examples of complex manifolds.** We give few examples of complex manifolds:

Riemman surfaces: Riemann surfaces are complex manifolds of complex dimension one. Recall that, topologically, oriented surfaces are classified by their *genus* g as g-holed tori. Because of $GL(1;\mathbb{C}) = CO(2;\mathbb{R})$, the choice of a(n almost-)complex structure is equivalent to the choice of a conformal class of Riemannian metrics. This is in fact the existence of isothermal coordinates.

compact complex surfaces: Compact complex surfaces are classified according to the Enriques-Kodaira-Siu classification [Enr49, Kod64, Kod68b, Kod68a, Kod66, Siu83]. (See [Nak84, BHPVdV04, Tel19].) It founds on the following invariants: *plurigenera* are $P_\ell := \dim_{\mathbb{C}} H^0(X;\ell K_X)$ varying $\ell \in \mathbb{N}$; the *Kodaira-Iitaka dimension* is

$$\mathrm{Kod}(X) := \limsup_{\ell\to+\infty} \frac{\log\dim H^0(X;\ell K_X)}{\log \ell},$$

(namely, the supremum of the ranks of the canonical maps $\Phi_\ell\colon X\setminus \mathrm{BaseLocus} \to \mathbb{P}H^0(X;\ell K_X)^*$;) the Hodge numbers $h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X)$ yield, in particular, the *irregularity* $q := h^{0,1}$, the *geometric genus* $p_g := h^{0,2}$, the *arithmetic genus* $p_a := p_g - q$. A complete classification of non-Kählerian surfaces is still missing, because of *surfaces of class VII*: they have $\mathrm{Kod}(X) = -\infty$ and $b_1 = 1$; when $b_2 = 0$, they are Hopf surfaces and Inoue(-Bombieri) surfaces [Bog76, Bog96, LYZ90, LYZ94, Tel94]; when $b_2 > 0$, they are Inoue-Hirzebruch surfaces, Enoki surfaces, Kato surfaces; the *Global Spherical Shell conjecture* states that minimal class VII surfaces with positive second Betti number are Kato surfaces, and works by Andrei Teleman [Tel05, Tel10, Tel17, Tel19] and references therein give positive evidences for the conjecture at least for b_2 small.

projective manifolds: Projective manifolds are compact complex submanifolds of $\mathbb{CP}^n := \left(\mathbb{C}^{n+1}\setminus\{0\}\right)/\left(\mathbb{C}\setminus\{0\}\right)$. By the Chow theorem [Cho49, Theorem V], see also Serre GAGA [Ser56], (stating the equivalence between the category of coherent algebraic sheaves on a complex projective variety and the category of coherent analytic sheaves on the corresponding analytic space,) they are in fact algebraic manifold, that is, they can be described as the zero set of finitely many homogeneous holomorphic polynomials.

complex tori: Consider a lattice Γ in $\mathbb{C}^n$, that is, a discrete subgroups of maximal rank $2n$ (*e.g.* $\Gamma = \mathbb{Z}\langle a\tau+b, c\tau+d\rangle$ in $\mathbb{C}$). Then the action of Γ over $\mathbb{C}^n$ is fixed-point-free and properly-discontinuous, whence $\mathbb{C}^n/\Gamma$ is a manifold endowed with an induced complex structure. It is homeomorphic to $\mathbb{R}^{2n}/\mathbb{Z}^{2n} = (S^1)^n$. One-dimensional tori are algebraic: their embeddings in the projective space are *elliptic curves*, namely, non-singular algebraic curves defined by equation of the form $y^2 = x^3 + ax + b$. The moduli space of complex structures (up to biholomorphisms) of 1-dimensional tori is parametrized by the elliptic modular function $J(\tau) := \frac{a\tau+b}{c\tau+d}$ taking values in the fundamental domain

$$\left\{ \begin{array}{c} |\tau| \geq 1 \\ -\frac{1}{2} < \Re(\tau) \leq \frac{1}{2} \\ -\frac{1}{2} < \Re(\tau) < 0 \Rightarrow |\tau| > 1 \end{array} \right\} .$$

spheres: The only spheres that admit almost-complex structures are S^2 and S^6 [BS53], see also *e.g.* [May99, Section 24.4]. Indeed, the Chern class of S^{2n} is $c_n = \chi = 2[S^{2n}]^\vee \in H^{2n}(S^{2n};\mathbb{Z})$, and it must be divisible by $(n-1)!$. This happens only for $n \leq 3$. For S^2 and S^6 we can construct explicitly an almost-complex structure by using quaternions and octonions. It is known that S^4 does not admit almost-complex structures. Any almost-complex structure on S^2 is clearly integrable. The octonionic almost-complex structure on S^6 is not integrable. It is not known whether S^6 admits integrable almost-complex structures, which is called *Hopf problem* [Hop48]; more in general, it is not known whether, in dimension $2n \geq 6$, there exist manifolds admitting almost-complex structures and with no complex structure, which is called the *Yau problem* [Gre93, Problem IV.52 at page 15]. In dimension 4, examples of 4-dimensional manifolds admitting almost-complex but no complex structures can be found in [VdV66]. It has been proved by C. LeBrun [LeB87] that there is no integrable almost-complex structures on S^6 that is compatible with the round metric; this has been generalized for metrics in a certain neighbourhood of the round metric by Bor and Hernández-Lamoneda [BHL99]. Some can be said about cohomological properties of hypothetical complex structures on the sphere [Gra97b, Uga00a, McH], in particular $h^{0,1} \geq 1$, and its double complex can be described. Clearly, a hypothetical complex structure X would be non-Kähler, not even Moishezon; in fact, its algebraic dimension $a(X) := \operatorname{tr}\deg_{\mathbb{C}} \mathcal{M}(X)$ where $\mathcal{M}(X)$ is the field of meromorphic functions is expected to be $a(X) = 0$ [CDP98]. Recently, there has been great interest around the problems, with possible directions proposed by G. Etesi [Ete15b, Ete15c, Ete15a] towards a positive answer, and by M. Atiyah [Ati16] towards a negative answer to the existence of a complex structure on $\mathbb{S}^6$.

locally-homogeneous manifolds of solvable Lie groups: Let G be a connected simply-connected Lie group, and let Γ be a co-compact discrete subgroup. Consider $X := \Gamma\backslash G$. If G is Abelian, then X is a torus. If G is nilpotent, then X is called *nilmanifold* [Mal49, page 278]. More in general, if G is solvable, then X is called *solvmanifold*. Taking an invariant complex structure on G with respect to left-translations (namely, a linear complex structure on the associated Lie algebra $\mathfrak{g}$ satisfying the integrability condition by Newlander-Nirenberg) induces a complex structure on X. Examples include: the *Hopf surface* $\mathbb{C}^2 \setminus \{0\} / \langle z \mapsto 2z \rangle$; the *Inoue surfaces*; the *Iwasawa manifold* $\mathrm{GL}\left(3; \mathbb{Z}\left[\sqrt{-1}\right]\right)\backslash \mathrm{GL}(3;\mathbb{C})$; the *Nakamura manifold*.

E.g., as for nilmanifolds: on the one hand, non-tori nilmanifolds admit no Kähler structure, [BG88, Theorem A], [Has89, Theorem 1, Corollary]; on the other hand, the study of geometric structures can be often reduced at the level of the associated Lie algebra, by averaging to invariant structures on the Lie group [FG04], and also cohomologies [Nom54, Hat60, Con06]. As for 6-dimensional nilmanifolds: V. V. Morozov [Mor58] classified of 6-dimensional nilpotent Lie algebras, up to isomorphism, in 34 classes; see also [Mag86], [Boc09, Table 15], [Gon98, §3]. S. M. Salamon [Sal01] proved that 18 classes out of these 34 admit a linear integrable complex structure, [Sal01, Theorem 3.1, Theorem 3.2, Theorem 3.3, Proposition 3.4]. The classification, up to equivalence, of these linear integrable complex structures has been completed thanks to: L. Ugarte and R. Villacampa [UV14]; A. Andrada, M. L. Barberis, and I. G. Dotti [ABDM11]; M. Ceballos, A. Otal, L. Ugarte, and R. Villacampa [COUV16].

LVMB manifolds: Founded on the idea of the Hopf and Calabi-Eckmann manifolds, López de Medrano and Verjovsky [LdMV97], Meersseman [Mee97, Mee00], and Bosio [Bos01] proposed a construction of compact complex manifolds as follows. They are leaf spaces of a foliation of $\mathbb{CP}^{n-1}$ given by suitable linear actions $\mathbb{C}^m \circlearrowleft \mathbb{C}^n$. The problem of embedding compact complex manifolds transversely to an algebraic foliation in a complex projective algebraic variety is related to a conjecture by Bogomolov [Bog96] and has been recently investigated in [DG14].

OT manifolds: Oeljeklaus-Toma manifolds [OT05] provide a family of examples of compact complex non-Kähler manifolds, generalizing Inoue-Bombieri surfaces and introduced as counterexamples to a conjecture by I. Vaisman concerning locally conformally Kähler metrics. Because of their construction using number fields techniques, many of their properties are encoded in the algebraic structure [OT05, Vul14, Dub14], and their class is well-behaved under such properties [Ver14, Ver13]. They generalize Inoue-Bombieri surfaces in class VII [Ino74, Bom73, Tri82], and they are in fact solvmanifolds [Kas13c].

deformations of complex structures: A natural way to construct new complex structures on a manifold is by deformations of a given complex structure.

Let B be a complex (respectively, differentiable) manifold. A family $\{X_t\}_{t\in B}$ of compact complex manifolds is said to be a *complex-analytic family of compact complex manifolds* if there exist a complex manifold $\mathcal{X}$ and a surjective holomorphic map $\pi\colon \mathcal{X} \to B$ such that

(i) $\pi^{-1}(t) = X_t$ for any $t \in B$, and

(ii) π is a proper holomorphic submersion.

A compact complex manifold X is said to be a *deformation* of a compact complex manifold Y if there exist a complex-analytic family $\{X_t\}_{t\in B}$ of compact complex manifolds, and $b_0, b_1 \in B$ such that $X_{b_0} = X_s$ and $X_{b_1} = X_t$.

The theory of deformations of complex manifolds has been started and developed by K. Kodaira, D. C. Spencer, L. Nirenberg, and M. Kuranishi [KS58, KS60, KNS58, Kur62]. As a reference, see, *e.g.*, [Kod05, MK06, Huy05].

I.2. The double complex of forms on a complex manifold

Let X be a differentiable manifold of even dimension $\dim X = 2n$. We consider J an almost-complex structure, (that is, an endomorphism $J \in \mathrm{End}(TX)$ such that $J^2 = -\mathrm{id}$). When it is integrable (*i.e.* $\mathrm{Nij}_J(x,y) := [x,y] + J[Jx,y] + J[x,Jy] - [Jx,Jy] = 0$ for any x, y), then we can actually construct holomorphic coordinates $(z^j = x^j + \sqrt{-1}y^j)_j$ on X thanks to the Newlander and Nirenberg theorem, and X is then a complex manifold. Let us focus on the integrable case.

We split the complexified tangent bundle in the eigenbundles of J:

$$\begin{aligned} TX \otimes \mathbb{C} &= T^{1,0}X \oplus T^{0,1}X \\ &= \mathrm{span}\left\{Z^j\right\}_j \oplus \mathrm{span}\left\{\bar{Z}^j\right\}_j \\ &= \mathrm{span}\left\{\frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - \sqrt{-1}\frac{\partial}{\partial y^j}\right)\right\}_j \\ &\quad \oplus \mathrm{span}\left\{\frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + \sqrt{-1}\frac{\partial}{\partial y^j}\right)\right\}_j , \end{aligned}$$

where $\{Z^j\}_j$ is a local frame for $T^{1,0}X$, that we can take as $Z^j = \frac{\partial}{\partial z^j}$ when the almost-complex structure is integrable. (In the non-integrable almost-complex case, we can still split the complexified tangent bundle in its $\mathbb{C}$-linear and anti-$\mathbb{C}$-linear part, but the local frame is not induced by holomorphic coordinates: in this case, integrability can be restated as $T^{1,0}X$ being involutive for the Lie bracket.)

Similarly, we decompose the complexified cotangent bundle into eigenbundles for the dual endomorphism $J\alpha := \alpha(J^t\_) = \alpha(-J^{-1}\_) = \alpha(J\_)$ (note that different notations may be used concerning the sign):

$$\begin{aligned} T^*X \otimes \mathbb{C} &= (T^{1,0}X)^* \oplus (T^{0,1}X)^* \\ &= \mathrm{span}\left\{\alpha^j\right\}_j \oplus \mathrm{span}\left\{\bar{\alpha}^j\right\}_j \\ &= \mathrm{span}\left\{dz^j = dx^j + \sqrt{-1}dy^j\right\}_j \oplus \mathrm{span}\left\{d\bar{z}^j = dx^j - \sqrt{-1}dy^j\right\}_j , \end{aligned}$$

where $\{\alpha^j\}_j$ is the local dual coframe for $(T^{1,0}X)^*$ of $\{Z^j\}_j$, that we can take as $\alpha^j = dz^j$ when the almost-complex is integrable. We get a decomposition for the bundle of forms

$$\wedge^k T^*X = \bigoplus_{p+q=k} \wedge^p (T^{1,0}X)^* \otimes \wedge^q (T^{0,1}X)^*,$$

where a smooth section of $\wedge^{p,q}X := \wedge^p(T^{1,0}X)^* \otimes \wedge^q(T^{0,1}X)^*$ is locally given by

$$\begin{aligned} &\sum_{\substack{i_1<\cdots<i_p \\ j_1<\cdots<j_q}} \varphi_{i_1,\ldots,i_p,j_1,\ldots,j_q} \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_p} \wedge \bar{\alpha}^{j_1} \wedge \cdots \wedge \bar{\alpha}^{j_q} \\ &= \sum_{\substack{i_1<\cdots<i_p \\ j_1<\cdots<j_q}} \varphi_{i_1,\ldots,i_p,j_1,\ldots,j_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}, \end{aligned}$$

for smooth functions $\varphi_{i_1,\ldots,i_p,j_1,\ldots,j_q}$, the second line holding when J is integrable.

Since d sends functions to $\wedge^{1,0}X \oplus \wedge^{0,1}X$, and 1-forms to $\wedge^{2,0}X \oplus \wedge^{1,1}X \oplus \wedge^{0,2}X$, since every differential form is locally a finite sum of decomposable differential forms, and by the Leibniz rule, the $\mathbb{C}$-linear extension of the exterior differential *a priori* splits into four components:

$$d = A + \partial + \overline{\partial} + \overline{A} \colon\ \wedge^{p,q} X \to \wedge^{p+2,q-1}X \oplus \wedge^{p+1,q}X \oplus \wedge^{p,q+1}X \oplus \wedge^{p-1,q+2}X.$$

When local holomorphic coordinates exist, we have in fact just

$$d = \partial + \overline{\partial} \colon\ \wedge^{p,q} X \to \wedge^{p+1,q}X \oplus \wedge^{p,q+1}X,$$

and the condition $d^2 = 0$ reads as

$$\partial^2 = \partial\overline{\partial} + \overline{\partial} = \overline{\partial}^2 = 0.$$

In fact, integrability of almost-complex structure J is equivalent to the vanishing of the components A and $\overline{A}$ of the exterior differential. In other words, for an integrable almost-complex structure, we have the double complex

$$\left(\wedge^{\bullet,\bullet}X, \partial, \overline{\partial}\right).$$

We will depict it as in Figure 1, where each square consists of (p,q)-forms, horizontal arrows represent the ∂ operator and vertical arrows represent the $\overline{\partial}$ operator.

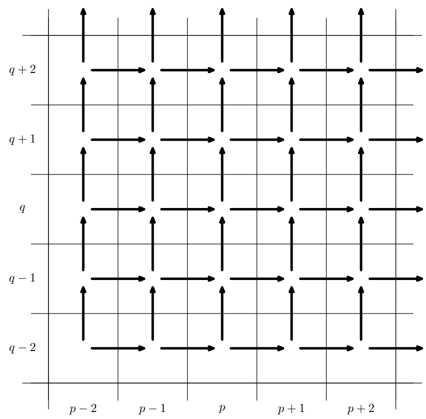


FIGURE 1. Double complex of forms of a complex manifold.

The double complex has some symmetries: conjugation yields a symmetry around the bottom-left/top-right diagonal; moreover, the duality given by any Hermitian metric is expected to yield a symmetry around the bottom-right/top-left diagonal.

We focus on special objects in the diagram. In the following, each point represent the 1-dimensional $\mathbb{C}$-vector spaces and arrows are isomorphisms:

- *zigzags* of length $\ell+1$, where $\ell+1$ is the number of dots, and $\ell \in \mathbb{N}$ counts the number of arrows:

$$\begin{tikzcd}
\cdots & \bullet & & & \\
& \bullet \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"'] & \bullet & & \\
& & \ddots \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"'] & \bullet & \\
& & & \bullet \arrow[u, "\overline{\partial}"] & \cdots
\end{tikzcd}$$

  in particular, zigzags of length one are called *dots*;
- *squares* of isomorphisms:

$$\begin{tikzcd}
\bullet \arrow[r, "\partial", "\simeq"'] & \bullet \\
\bullet \arrow[u, "\overline{\partial}", "\simeq"'] \arrow[r, "\simeq", "\partial"'] & \bullet \arrow[u, "\simeq", "\overline{\partial}"']
\end{tikzcd}$$

Note that the French school prefers to consider instead $(\wedge^{\bullet,\bullet}X, d, d^c)$ for the *conjugate differential*

$$d^c \colon \wedge^{\bullet} X \to \wedge^{\bullet+1}X, \qquad d^c := \frac{1}{2}\, J^{-1} d J = -\frac{\sqrt{-1}}{2}(\partial - \overline{\partial}),$$

so that $dd^c = \sqrt{-1}\, \partial\overline{\partial}$.

The following folklore result cited by Greg Kuperberg in the MathOverflow discussion at http://mathoverflow.net/questions/25723/ is originally attributed to Mikhail Khovanov [Khote], and a complete proof has been finally given by Jonas Stelzig in his interesting PhD Thesis:

Theorem I.2.1 (Stelzig [Ste18b])**.** *Every bounded double complex can be decomposed as direct sum of squares and zigzags, whose multiplicity is uniquely determined.*

For example, once removed the infinite squares and the arrows arising from symmetries, the double complex associated to a hypothetical complex structure on the 6-dimensional sphere $\mathbb{S}^6$ should be as in Figure 2, where the labels count the number of respective objects and α, $h^{0,2}$, β, $h^{1,0}$, $h^{1,1}$ are unknown non-negative integers. This example is constructed by using the results in [Gra97a, Uga00b] on the Dolbeault cohomology and the Frölicher spectral sequence of a hypothetical complex structure on the six sphere.

Remark I.2.2. Note that, if E is a holomorphic vector bundle over the complex manifold X, then there is a well-defined differential operator

$$\overline{\partial} \colon \wedge^{0,q}(X; E) \to \wedge^{0,q+1}(X; E),$$

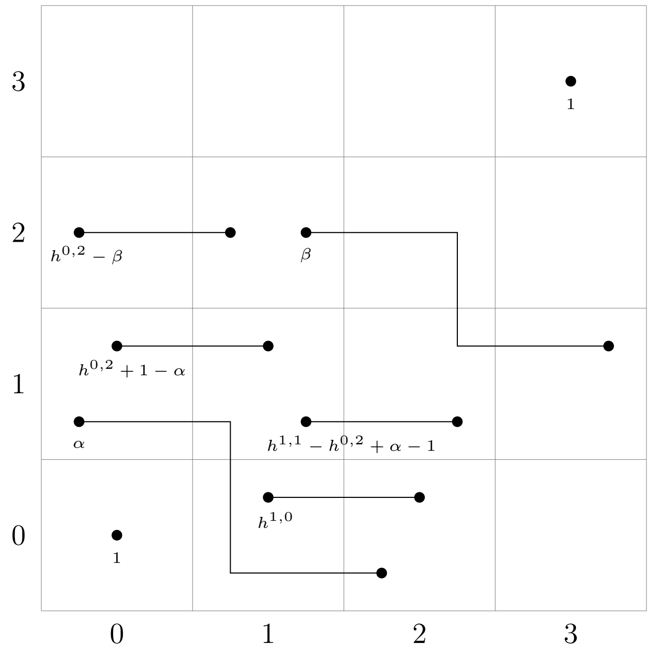


FIGURE 2. The double complex of forms for a hypothetical complex structure on the six-sphre.

where $\wedge^{0,q}(X;E)$ denotes the space of sections of smooth forms with value in E, that is, the section of the bundle $(\wedge^{0,q}T^*X \otimes \mathbb{C}) \otimes E$. Then we have the differential complex $(\wedge^{0,\bullet}(X;E), \overline{\partial})$.

I.3. Cohomologies of complex manifolds

In the double complex above, we can just forget the horizontal arrows, and consider, for any column p, the single complex $(\wedge^{p,\bullet}X, \overline{\partial})$, see Figure 3. Its cohomology is the *Dolbeault cohomology*

$$H^{\bullet,\bullet}_{\overline{\partial}}(X) = \frac{\ker \overline{\partial}}{\operatorname{im} \overline{\partial}}.$$

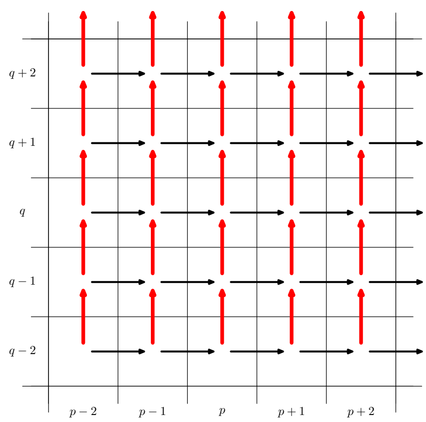


FIGURE 3. Dolbeault cohomology.

The following statements provide alternative points of view and further tools, respectively, sheaf-theoretic and analytic.

Theorem I.3.1 (Dolbeault [Dol53]). *On a complex manifold X,*

$$H^{p,q}_{\overline{\partial}}(X) = H^q(X; \Omega^p_X)$$

where Ω^p_X denotes the sheaf of germs of holomorphic p-forms.

(For the following, please refer to Section III.10.1 for notation and details.)

Theorem I.3.2 (Hodge [Hod89]). *On a compact complex manifold X endowed with a Hermitian metric g, consider the adjoint operator $\overline{\partial}^*$ of $\overline{\partial}$ with respect to the L^2 pairing induced by g on $\wedge^{\bullet,\bullet}X$. Define the second-order self-adjoint elliptic differential operator $\overline{\square} := [\overline{\partial}, \overline{\partial}^*] = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$. Then*

$$H^{\bullet,\bullet}_{\overline{\partial}}(X) \simeq \ker \overline{\square}.$$

In particular, $h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X) < +\infty$, and depends upper-semi-continuously under small deformations of X.

Once fixed a Hermitian metric g and its volume vol, we have the $\mathbb{C}$-linear Hodge-star-operator $*\colon \wedge^{p,q} X \to \wedge^{n-q,n-p} X$, where $\dim_{\mathbb{C}} X = n$, such that $\alpha \wedge *\bar{\beta} = \langle \alpha | \beta \rangle \operatorname{vol}$. Then $\overline{\partial} = - * \partial *$. This yields to the following symmetry:

Theorem I.3.3 (Serre [Ser55]). *On a compact complex manifold X of complex dimension n, the Hodge-star-operator of a Hermitian metric induces the isomorphism*

$$H^{p,q}_{\overline{\partial}}(X) \simeq \overline{H^{n-p,n-q}_{\overline{\partial}}(X)}.$$

In general, there is no natural map connecting the Dolbeault and the de Rham cohomologies. But we have the following comparison:

Theorem I.3.4 (Frölicher [Frö55]). *On a compact complex manifold X, the filtration $F^p(\wedge^k X \otimes \mathbb{C}) := \bigoplus_{\substack{r+s=k \\ r \geq p}} \wedge^{r,s} X$ induce a spectral sequence with first page*

$$E_1^{p,q} = H^{p,q}_{\overline{\partial}}(X) \Rightarrow H^{p+q}_{dR}(X; \mathbb{C})$$

and converging to the de Rham cohomology. In particular, for any $k \in \mathbb{N}$,

$$\sum_{p+q=k} h^{p,q} \geq b_k$$

where b_k denotes the kth Betti number.

An explicit description of the Frölicher spectral sequence can be found in [CFGU00b, Theorem 1, Theorem 3]:

$$E_r^{p,q} \simeq \frac{\mathcal{X}_r^{p,q}}{\mathcal{Y}_r^{p,q}},$$

where, for $r = 1$,

$$\mathcal{X}_1^{p,q} := \left\{ \alpha \in \wedge^{p,q} X \ : \ \overline{\partial}\alpha = 0 \right\}, \qquad \mathcal{Y}_1^{p,q} := \overline{\partial} \wedge^{p,q-1} X,$$

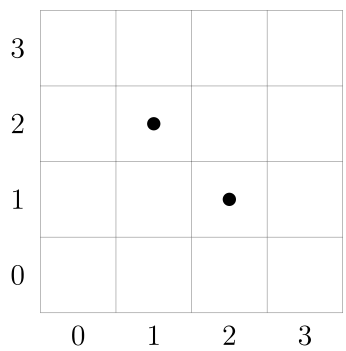
and, for $r \geq 2$,

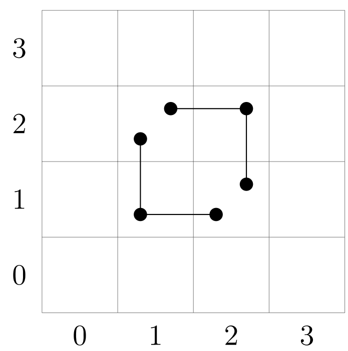
$$\begin{aligned}
\mathcal{X}_r^{p,q} \quad := \quad & \big\{\alpha^{p,q} \in \wedge^{p,q}X \;:\; \overline{\partial}\alpha^{p,q} = 0 \text{ and, for any } i \in \{1, \ldots, r-1\}, \\
& \text{there exists } \alpha^{p+i,q-i} \in \wedge^{p+i,q-i}X \\
& \text{such that } \partial\alpha^{p+i-1,q-i+1} + \overline{\partial}\alpha^{p+i,q-i} = 0\big\}\,, \\
\mathcal{Y}_r^{p,q} \quad := \quad & \big\{\partial\beta^{p-1,q} + \overline{\partial}\beta^{p,q-1} \in \wedge^{p,q}X \;:\; \text{for any } i \in \{2, \ldots, r-1\}, \\
& \text{there exists } \beta^{p-i,q+i-1} \in \wedge^{p-i,q+i-1}X \\
& \text{such that } \partial\beta^{p-i,q+i-1} + \overline{\partial}\beta^{p-i+1,q+i-2} = 0 \text{ and } \overline{\partial}\beta^{p-r+1,q+r-2} = 0\big\}\,,
\end{aligned}$$

and, for any $r \geq 1$, the map $d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ is given by

$$d_r[\alpha^{p,q}] := [\partial\alpha^{p+r-1,q-r+1}].$$

We rephrase again this issue by noticing that the Dolbeault and de Rham cohomologies do not suffice, in general, for detecting the complete structure of the double complex. For example, zigzags of odd length do not contribute to the difference between Dolbeault and de Rham cohomology. This means that symmetric zigzags of odd length cannot be detected by the Frölicher spectral sequence. For example, the following diagrams have the same de Rham and Dolbeault cohomologies:





The above diagrams differ as for the number of corners. Whence we get the need for having an invariant that counts the corners. Let us introduce the *Bott-Chern cohomology* [BC65] (see Figure 4) and its "dual" the *Aeppli cohomology* [Aep65] (see Figure 5)

$$H^{\bullet,\bullet}_{BC}(X) := \frac{\ker\partial \cap \ker\overline{\partial}}{\operatorname{im}\partial\overline{\partial}}, \qquad H^{\bullet,\bullet}_{A}(X) := \frac{\ker\partial\overline{\partial}}{\operatorname{im}\partial + \operatorname{im}\overline{\partial}}$$

in order to solve the issue of no natural map connecting the de Rham and the Dolbeault cohomologies.

In the same spirit as de Rham cohomology is related to Maxwell equations of electromagnetism, then Bott-Chern-Aeppli cohomologies play a role in Type II String Theory [TY14].

Also for Bott-Chern and Aeppli cohomologies, we have sheaf-theoretic and analytic interpretations.

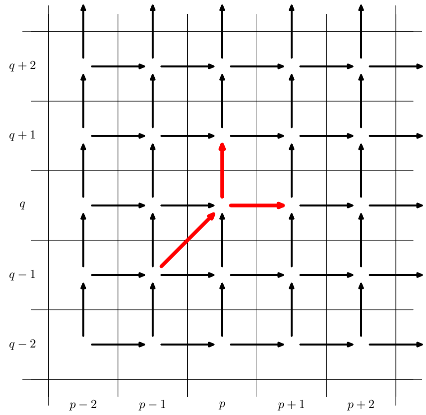


FIGURE 4. Bott-Chern cohomology.

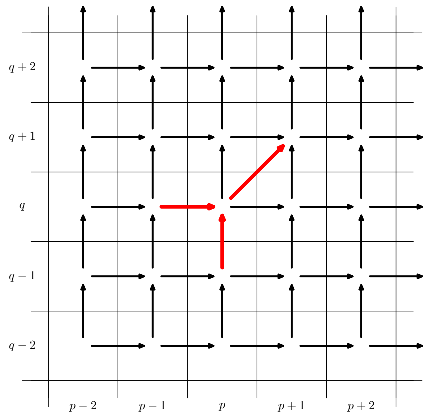


FIGURE 5. Aeppli cohomology.

Theorem I.3.5 (Schweitzer [Sch07]). *On a complex manifold X, consider the complex $\mathcal{L}^\bullet_{p,q}$ of sheaves*

$$\cdots \stackrel{\mathrm{pr}\circ d}{\to} \bigoplus_{\substack{r+s=p+q-3\\ r<p,\, s<q}} \mathcal{A}^{r,s} \stackrel{\mathrm{pr}\circ d}{\to} \bigoplus_{\substack{r+s=p+q-2\\ r<p,\, s<q}} \mathcal{A}^{r,s} \stackrel{\partial\overline{\partial}}{\to} \bigoplus_{\substack{r+s=p+q\\ r\geq p,\, s\geq q}} \mathcal{A}^{r,s} \stackrel{d}{\to} \bigoplus_{\substack{r+s=p+q+1\\ r\geq p,\, s\geq q}} \mathcal{A}^{r,s} \to \cdots,$$

which is quasi-isomorphic to the shifting of the complex $\mathcal{B}^\bullet_{p,q}$ of sheaves

$$\mathbb{C} \stackrel{(+,-)}{\to} \mathcal{O} \oplus \overline{\mathcal{O}} \to \Omega^1 \oplus \overline{\Omega}^1 \to \cdots \to \Omega^{p-1} \oplus \overline{\Omega}^{p-1} \to \overline{\Omega}^p \to \cdots \to \overline{\Omega}^{q-1} \to \overline{\Omega}^q \to 0. \tag{2}$$

Then

$$H^{p,q}_{BC}(X) \simeq \mathbb{H}^{p+q-1}(X; \mathcal{L}^\bullet_{p,q}) \simeq \mathbb{H}^{p+q}(X; \mathcal{B}^\bullet_{p,q}),$$
$$H^{p,q}_{A}(X) \simeq \mathbb{H}^{p+q}(X; \mathcal{L}^\bullet_{p+1,q+1}) \simeq \mathbb{H}^{p+q+1}(X; \mathcal{B}^\bullet_{p+1,q+1}).$$

One can now define the *integral Bott-Chern cohomology* [Sch07] as

$$H^{p,q}_{BC}(X; \mathbb{Z}) := \mathbb{H}^{p+q}(X; \mathbb{B}^\bullet_{\mathbb{Z}(p)})$$

where $B^\bullet_{\mathbb{Z}(p)}$ is defined as in (2) by replacing $\mathbb{C}$ with $\mathbb{Z}(p) := (2\pi\sqrt{-1})^p\mathbb{Z}$.

Theorem I.3.6 (Kodaira-Spencer [KS60], Schweitzer [Sch07])**.** *On a compact complex manifold X endowed with a Hermitian metric g, consider*

$$\tilde{\Delta}_{BC} := \left(\partial\overline{\partial}\right)\left(\partial\overline{\partial}\right)^* + \left(\partial\overline{\partial}\right)^*\left(\partial\overline{\partial}\right) + \left(\overline{\partial}^*\partial\right)\left(\overline{\partial}^*\partial\right)^* + \left(\overline{\partial}^*\partial\right)^*\left(\overline{\partial}^*\partial\right) + \overline{\partial}^*\overline{\partial} + \partial^*\partial$$

and

$$\tilde{\Delta}_{A} := \partial\partial^* + \overline{\partial}\overline{\partial}^* + \left(\partial\overline{\partial}\right)^*\left(\partial\overline{\partial}\right) + \left(\partial\overline{\partial}\right)\left(\partial\overline{\partial}\right)^* + \left(\overline{\partial}\partial^*\right)^*\left(\overline{\partial}\partial^*\right) + \left(\overline{\partial}\partial^*\right)\left(\overline{\partial}\partial^*\right)^*,$$

which are 4th order self-adjoint elliptic differential operators. Then

$$H^{\bullet,\bullet}_{BC}(X) \simeq \ker \Delta_{BC}, \qquad H^{\bullet,\bullet}_{A}(X) \simeq \ker \Delta_{A}.$$

In particular, $h^{p,q}_{BC}(X) := \dim H^{p,q}_{BC}(X) < +\infty$ and $h^{p,q}_{A}(X) := \dim H^{p,q}_{A}(X) < +\infty$, and depends upper-semi-continuously under small deformations of X.

Of course, the conjugation induces the symmetries

$$H^{p,q}_{BC}(X) \simeq \overline{H^{q,p}_{BC}(X)}, \qquad H^{p,q}_{A}(X) \simeq \overline{H^{q,p}_{A}(X)}.$$

Moreover, the following symmetry holds:

Theorem I.3.7 (Schweitzer [Sch07])**.** *On a compact complex manifold X, endowed with a Hermitian metric, the $\mathbb{C}$-linear Hodge-star-operator induces the isomorphism*

$$*\colon H^{p,q}_{BC}(X) \stackrel{\simeq}{\to} H^{n-q,n-p}_{A}(X).$$

Note that it mixes up the two cohomologies and, in general, it is not internal to just Bott-Chern cohomology, see Theorem II.8.2.

I.4. Cohomologies of quotients of Lie groups

In this section, we focus on "model" for cohomologies, in the sense of rational homotopy theory, see *e.g.* [Sul76, FHT15, FOT08]. More precisely, we reduce the computation of cohomologies for some classes of solvmanifolds $\Gamma\backslash G$ to the sub-complex of *invariant forms*, namely, forms whose lift to the covering Lie group is invariant under the left-translations. We notice that this is the same as the complex $\wedge^\bullet\mathfrak{g}^\vee$ constructed over the Lie algebra.

I.4.1. **Cohomologies of tori.** As a first step, we prove that the de Rham cohomology of tori is recovered by invariant forms. More in general:

Theorem I.4.1 (see *e.g.* [FOT08, Theorem 1.28])**.** *Let M be a compact differentiable manifold with an left-action of a compact connected Lie group G. Then the de Rham cohomology of M is isomorphic to the cohomology of the complex of G-invariant forms.*

Proof. Denotes by $i\colon (\wedge^\bullet M)^G \to \wedge^\bullet M$ the inclusion. We have to prove that the induced map $H(i)\colon H^\bullet(M;\mathbb{R})^G \to H^\bullet(M;\mathbb{R})$ is an isomorphism. We first notice that the space of G-invariant form is actually a sub-complex $((\wedge^\bullet M)^G, d)$ of the complex of forms $(\wedge^\bullet X, d)$, whose cohomology we denote by $H^\bullet(M;\mathbb{R})^G$. We then fix a bi-invariant volume form dg on

G, thanks to [Mil76], see *e.g.* [FOT08, Proposition 1.29]. We consider the following average map:

$$\wedge^\bullet M \ni \alpha \stackrel{\mu}{\mapsto} \int_G g^* \alpha \, dg \in (\wedge^\bullet M)^G .$$

It is easy to check that $\mu(\alpha)$ is actually G-invariant. Moreover, $\mu(\alpha) = \alpha$ if and only if α is itself G-invariant. Rewrite this as $\mu \circ i = \mathrm{id}$. We also notice that $d\mu = \mu d$, whence μ induces a map in cohomology: $H(\mu)\colon H^\bullet(M \, \mathbb{R}) \to H^\bullet(M;\mathbb{R})^G$. In fact, by $\mu \circ i = \mathrm{id}$, we get $H(\mu) \circ H(i) = \mathrm{id}$ whence $H(i)$ is injective. It remains to prove that $H(i)$ is surjective. This follows by the fact that we can reduce integration to a neighbourhood of the identity in G, in particular, to a contractible neighbourhod. Indeed it follows that μ is quasi-isomorphic to the identity, whence $H(i)$ is surjective. □

Remark I.4.2. Note that the same does not hold true, in general, for the Dolbeault cohomology, see also [Les93, Akh97].

I.4.2. **Cohomologies of nilmanifolds.** We prove now the Nomizu theorem:

Theorem I.4.3 (Nomizu [Nom54])**.** *Let $X = \Gamma\backslash N$ be a nilmanifold, namely, a compact quotiente of a connected simply-connected Lie group N by a cocompact discrete subgroup Γ. Consider the Lie algebra $\mathfrak{n}$ of N, and the sub-complex of invariant forms $(\wedge^\bullet \mathfrak{n}^\vee, d) \hookrightarrow (\wedge^\bullet X, d)$. Then the inclusion is a quasi-isomorphism.*

Proof. The idea is the following, up to many technical details that we are deliberately avoiding. We write X as a tower of torus-bundles over low-dimensional nilmanifolds. We prove the statement for induction on the dimension of X. The base step corresponds to tori, already considered in the previous statement. The induction step founds on the following. For a fibre bundle

$$\pi\colon E \stackrel{F}{\to} B,$$

we consider the *Leray-Serre spectral sequence*. It is induced by the filtration

$$F^p \wedge^{p+q} E := \{\omega \in F^{p+q}E \;:\; \omega(v_1, \ldots, v_{p+q}) = 0 \text{ whenever } v_{i_1}, \ldots, v_{i_{p+1}} \in \ker d\pi\}.$$

Then

$$E_2^{p,q} = H^p(B; \underline{H}^q(F)) \Rightarrow H^{p+q}(E).$$

The induction hypothesis allows to compare the second pages of this spectral sequence, and to the corresponding one at the level of Lie algebras, whence also the last pages are isomorphic. □

Remark I.4.4. Similar results holds for the Dolbeault cohomology [Sak76, CFGU00a, CF01, Rol09, Rol11], and for the Bott-Chern cohomology [Ang13, AK17a], whenever we can replace the tower of torus-bundles by a tower of *holomorphic complex torus-bundles*. This happens for example for almost any invariant complex structure on 6-dimensional nilmanifolds, here including holomorphically-parallelizable Iwasawa [Sch07] and Nakamura manifolds. In other words, this means that the inclusion of the double complex of invariant forms into the whole double complex of differential forms is an E_1*-isomorphism* in the sense of Stelzig [Ste18b].

As a consequence, also Bott-Chern and Aeppli cohomologies are recovered by invariant forms [Ang13, AK17a, Ste18b].

I.4.3. **Cohomologies of special solvmanifolds.** As for the de Rham cohomology of solvmanifolds, when they are *completely-solvable* (in the sense that any $\mathrm{ad}_x \in \mathrm{End}(\mathfrak{g})$ has only real eigenvalues), then a result similar to the above Nomizu's theorem holds by [Hat60]. But in general left-invariant forms are usually not enough to recover the whole de Rham cohomology: see, for example, the non-completely-solvable solvmanifold provided in [dBT06, §3, §4, Corollary 4.2]. See also results in [Mos54, Gua07, CF11, Kas13a, Kas12a, CFK13, Kas13b, Kas12a, Kas14, Kas12b, Kas15].

We briefly summarize here some results concerning the Dolbeault cohomology of a class of solvmanifolds[Kas13a, Kas13b, AK17a], recalling the results on the computations of the cohomologies of the Nakamura manifold as an example. Even if we can not use the double complex of invariant forms, we can reduce in some case the computation of Dolbeault cohomology to a finite-dimensional complex.

In the following result, by *splitting-type* in the sense of [Kas13b] we mean a solvmanifold $X = \Gamma\backslash G$ endowed with a G-left-invariant complex structure J where G is a semi-direct product $\mathbb{C}^n \ltimes_\phi N$ satisfying the following assumptions:

(1) N is a connected simply-connected $2m$-dimensional nilpotent Lie group endowed with an N-left-invariant complex structure J_N;
(2) for any $t \in \mathbb{C}^n$, one has that $\phi(t) \in \mathrm{GL}(N)$ is a holomorphic automorphism of N with respect to J_N;
(3) ϕ induces a semi-simple action on the Lie algebra of N.

Theorem I.4.5 ([Kas13b, Corollary 4.2], [Kas12a, Corollary 6.2]). *Let $\Gamma\backslash G$ be a solvmanifold endowed with a G-left-invariant complex structure. Suppose that*

- *either $\Gamma\backslash G$ is of splitting-type, with the nilpotent factor N such that the inclusion $\left(\wedge^{\bullet,\bullet}\mathfrak{n}^*,\, \overline{\partial}\right) \hookrightarrow \left(\wedge^{\bullet,\bullet}N,\, \overline{\partial}\right)$ is a quasi-isomorphism, where $\mathfrak{n}$ is the Lie algebra of N,*
- *or $\Gamma\backslash G$ is holomorphically parallelizable.*

Then there exists a finite-dimensional sub-complex $B_\Gamma^{\bullet,\bullet}$ of the Dolbeault complex $\left(\wedge^{\bullet,\bullet}X,\, \partial,\, \overline{\partial}\right)$ such that the inclusion

$$\left(B_\Gamma^{\bullet,\bullet},\, \overline{\partial}\right) \hookrightarrow \left(\wedge^{\bullet,\bullet}\, \Gamma\backslash G\,,\, \overline{\partial}\right)$$

is a quasi-isomorphism.

Remark I.4.6 ([Kas13b]). In the case of solvmanifolds of splitting type, the sub-complex $B_\Gamma^{\bullet,\bullet}$ in the previous theorem is constructed as follows, see [Kas13b].

Consider the standard basis $\{X_1, \ldots, X_n\}$ of $\mathbb{C}^n$, and consider a basis $\{Y_1, \ldots, Y_m\}$ of the im-eigen-space $\mathfrak{n}^{1,0}$ of $J_N \in \mathrm{Aut}(\mathfrak{n})$ such that the induced action ϕ on $\mathfrak{n}^{1,0}$ is represented by $\phi = \mathrm{diag}\,(\alpha_1, \ldots, \alpha_m)$ for $\alpha_1 \in \mathrm{Hom}(\mathbb{C}^n; \mathbb{C}^*), \ldots, \alpha_m \in \mathrm{Hom}(\mathbb{C}^n; \mathbb{C}^*)$ characters of $\mathbb{C}^n$. Let $\left\{x_1, \ldots, x_n, \alpha_1^{-1}y_1, \ldots, \alpha_m^{-1}y_m\right\}$ be the basis of $\wedge^{1,0}\mathfrak{g}_{\mathbb{C}}^*$ which is dual to $\{X_1, \ldots, X_n, \alpha_1 Y_1, \ldots, \alpha_m Y_m\}$.

By [Kas13b, Lemma 2.2], for any $j \in \{1, \ldots, m\}$, there exist unique unitary characters β_j and γ_j of $\mathbb{C}^n$ such that $\alpha_j \beta_j^{-1}$ and $\bar{\alpha}_j \gamma_j^{-1}$ are holomorphic. (By shortening, *e.g.*, $\alpha_I := \alpha_{i_1} \cdot \cdots \cdot \alpha_{i_k}$ for a multi-index $I = (i_1, \ldots, i_k)$,) define $B_\Gamma^{\bullet,\bullet} \subset \wedge^{\bullet,\bullet} \Gamma\backslash G$, for $(p,q) \in \mathbb{Z}^2$, as

$$\begin{aligned} B_\Gamma^{p,q} \;&:=\; \mathbb{C}\left\langle x_I \wedge \left(\alpha_J^{-1}\beta_J\right) y_J \wedge \bar{x}_K \wedge \left(\bar{\alpha}_L^{-1}\gamma_L\right) \bar{y}_L \right. \\ &\qquad \left. :\; |I|+|J|=p \text{ and } |K|+|L|=q \text{ such that } \left(\beta_J\gamma_L\right)\lfloor_\Gamma = 1 \right\rangle \;. \end{aligned}$$

The previous results allow to compute explicitly the de Rham, Dolbeault, and Bott-Chern cohomologies of the completely-solvable Nakamura manifold and of the holomorphically parallelizable Nakamura manifold, [Kas13b, AK17a].

Example I.4.7 (The completely-solvable Nakamura manifold, [Kas13b, Example 1], [AK17a, Example 2.17])**.** The completely-solvable Nakamura manifold, firstly studied by I. Nakamura in [Nak75, page 90], is an example of a cohomologically Kähler non-Kähler solvmanifold, [dAFdLM92], [FMS03, Example 3.1], [dBT06, §3].

Consider the group

$$G \;:=\; \mathbb{C} \ltimes_\phi \mathbb{C}^2 \;, \qquad \text{where} \qquad \phi\left(x+\sqrt{-1}\,y\right) \;:=\; \left(\begin{array}{cc} \exp(x) & 0 \\ 0 & \exp(-x) \end{array}\right) \in \mathrm{GL}\left(\mathbb{C}^2\right) \;.$$

For some $a \in \mathbb{R}$, the matrix $\left(\begin{array}{cc} \exp(x) & 0 \\ 0 & \exp(-x) \end{array}\right)$ is conjugate to an element of $\mathrm{SL}(2;\mathbb{Z})$. Hence there exists a discrete co-compact subgroup

$$\Gamma \;:=\; \left(a\,\mathbb{Z} + b\,\sqrt{-1}\,\mathbb{Z}\right) \ltimes_\phi \Gamma_{\mathbb{C}^2}$$

of G, where $\Gamma_{\mathbb{C}^2}$ is a lattice of $\mathbb{C}^2$. The *completely-solvable Nakamura manifold* is the completely-solvable solvmanifold $X := \Gamma\backslash G$.

Consider holomorphic coordinates $\{z_1,\, z_2,\, z_3\}$ on X, where $\left\{z_1 := x+\sqrt{-1}\,y\right\}$ is the holomorphic coordinate on $\mathbb{C}$ (as a matter of notations, we shorten, for example, $\exp(-z_1)dz_{12\bar{1}} := \exp(-z_1)dz_1 \wedge dz_2 \wedge d\bar{z}_1$).

By A. Hattori's theorem, [Hat60, Corollary 4.2], the de Rham cohomology of $\Gamma\backslash G$ does not depend on Γ and can be computed using just G-left-invariant forms on $\Gamma\backslash G$; in Table 1, we list the harmonic representatives of the de Rham cohomology classes with respect to the G-left-invariant Hermitian metric $g := dz_1 \odot d\bar{z}_1 + \exp(-z_1-\bar{z}_1) dz_2 \odot d\bar{z}_2 + \exp(z_1+\bar{z}_1) dz_3 \odot d\bar{z}_3$.

As regards Dolbeault cohomology and Bott-Chern cohomology, they depend on the lattice, see [Kas13b, AK17a]. In particular, following [Kas13b, Example 1], one has to distinguish three different cases:

- ***(i):*** $b \in (2\,\mathbb{Z}) \cdot \pi$;
- ***(ii):*** $b \in (2\,\mathbb{Z}+1) \cdot \pi$;
- ***(iii):*** $b \notin \mathbb{Z} \cdot \pi$.

The Dolbeault cohomology of the completely-solvable Nakamura manifold was computed in [Kas13b, Example 5.1] by using Theorem I.4.5. The Bott-Chern cohomology of the completely-solvable Nakamura manifold was computed in [AK17a, Example 2.17], and is summarized in Table 2, see [AK17a, Table 4, Table 5, Table 3].

Finally, Table 3 summarizes the dimensions of the de Rham, Dolbeault, and Bott-Chern cohomologies of the completely-solvable Nakamura manifold, [Kas13b, Example 1], [AK17a, Example 2.17], see [AK17a, Table 6].

In particular, note that the completely-solvable Nakamura manifold in case *(iii)* satisfies the $\partial\overline{\partial}$-Lemma (compare also [Kas13b, Kas14]).

Lecture II. **Cohomological decomposition**

In this Lecture, we review some relation between the cohomologies introduced above, so to relate holomorphic and topological properties of complex manifolds. In particular, we introduce the notion of *$\partial\overline{\partial}$-Lemma property* as a cohomological decomposition, which holds in particular on compact Kähler manifolds; we give a numerical characterization of this property in terms of the Betti numbers and the dimensions of the Bott-Chern cohomology; and we study the behaviour of this property under deformations or modifications of the complex structures. A similar theory can be formulated for compact symplectic manifolds (the other side of Kähler geometry) and framed in the more general context of generalized-complex geometry. Finally, we will review a topological consequence of the $\partial\overline{\partial}$-Lemma property: if it holds, then the rational homotopy type of the complex manifold is a formal consequence of its de Rham cohomology.

References for this Lecture are [DGMS75, AT13, ASTT17, RYY17, Ste19, Gua04, Gua11, Cav05, AT15a].

II.5. Cohomological decomposition on complex manifolds: the $\partial\overline{\partial}$-Lemma

II.5.1. **The $\partial\overline{\partial}$-Lemma property.** Let X be a compact complex manifold, and let us consider the double complex $\left(\wedge^{\bullet,\bullet}X, \partial, \overline{\partial}\right)$ and its associated de Rham, Dolbeault, Bott-Chern and Aeppli cohomologies. The identity map induces the following natural maps:

$$\begin{tikzcd}
 & H^{\bullet,\bullet}_{BC}(X) \arrow[dl] \arrow[d] \arrow[dr] & \\
H^{\bullet,\bullet}_{\partial}(X) \arrow[r, Rightarrow] \arrow[dr] & H^{\bullet}_{dR}(X;\mathbb{C}) \arrow[d] & H^{\bullet,\bullet}_{\overline{\partial}}(X) \arrow[l, Rightarrow] \arrow[dl] \\
 & H^{\bullet,\bullet}_{A}(X) &
\end{tikzcd} \tag{3}$$

so allowing to compare the topological information contained in the de Rham cohomology and the holomorphic contents of Dolbeault and Bott-Chern cohomologies. Here, we just depicted $H^{\bullet,\bullet}_{\overline{\partial}}(X) \Rightarrow H^{\bullet}_{dR}(X;\mathbb{C})$ for the Frölicher spectral sequence: we recall that it does not yield a natural map in cohomology, but just allows to compare the dimension via the inequality $\sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X) \geq b_k(X)$.

Straightforward computations, which take advantage of ∂ and $\overline{\partial}$ having different degree, yield the following. It defines the notion of manifolds satisfying the $\partial\overline{\partial}$-Lemma:

Proposition II.5.1 (Deligne-Griffiths-Morgan-Sullivan [DGMS75])**.** *On a compact complex manifold, if the map $H^{\bullet,\bullet}_{BC}(X) \to H^{\bullet,\bullet}_{A}(X)$ is injective, then any map in diagram (3) is an isomorphism. In this case, we say that X* satisfies the $\partial\overline{\partial}$-Lemma.

Another equivalent restatement will be clearer in few minutes:

Theorem II.5.2 (Deligne-Griffiths-Morgan-Sullivan [DGMS75])**.** *A compact complex manifold X satisfies the $\partial\overline{\partial}$-Lemma if and only if its double complex is a sum of dots and squares, if and only if the Hodge-Frölicher spectral sequence degenerates at the first page and the natural filtration induces a Hodge structure of weight k on $H^k_{dR}(X;\mathbb{C})$ (that is, $H^k_{dR}(X;\mathbb{C}) = \bigoplus_{p+q=k} F^p H^k_{dR} \oplus \bar{F}^q H^k_{dR}$).*

We already recalled the Frölicher spectral sequence comparing the dimension of Dolbeault cohomology and the Betti numbers in Theorem I.3.4. The following provides an analogue of the Frölicher inequality for the Bott-Chern cohomology, and it can be interpreted as a quantitative characterization of the $\partial\overline{\partial}$-Lemma property (an unnatural isomorphism forcing a natural one).

Theorem II.5.3 (Angella-Tomassini [AT13])**.** *On a compact complex manifold X, we have the* inequality à la Frölicher

$$\sum_{p+q=k} \left(h^{p,q}_{BC} + h^{p,q}_{A}\right) \geq 2\, b_k. \tag{4}$$

Moreover, the equality holds for any k if and only if X satisfies the $\partial\overline{\partial}$-Lemma.

Idea of the proof. Let us understand the heuristic of the proof with the help of a concrete example. The following diagram represents (up to some unneccessary squares) the double complex of forms on the Iwasawa manifold

$$X := \mathbb{Z}[\sqrt{-1}]^3 \Big\backslash \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ & 1 & z_2 \\ & & 1 \end{pmatrix} \;:\; z_1, z_2, z_3 \in \mathbb{C} \right\},$$

thanks to works by Nomizu and Sakane [Nom54, Sak76, Sch07, Ang13].

It is easy to see that the squares do not contribute to any cohomology. And that the Bott-Chern cohomology counts the corners possibly having ingoing arrows, except for the squares:

$$\begin{tikzcd} \circ \arrow[r, dashed] & \bullet \\ & \circ \arrow[u, dashed] \end{tikzcd}$$

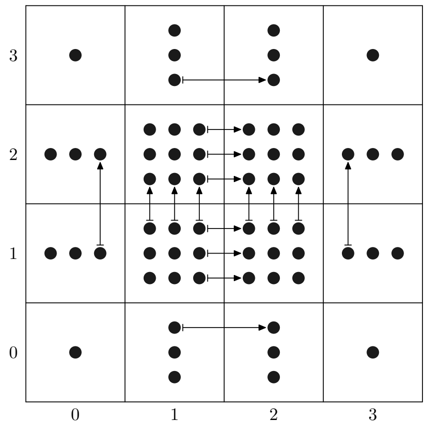
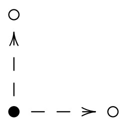
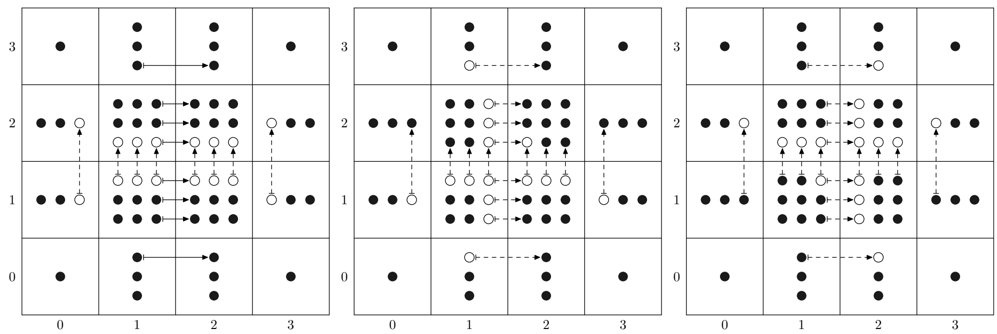


FIGURE 6. The double complex of (left-invariant) forms on the Iwasawa manifold.

Dually, Aeppli cohomology counts the corners possibly having outgoing arrows, except for the squares:



The following Figure 7 summarizes how to compute the cohomologies of the Iwasawa manifold, and the final result can be found in Table 4, also for the small deformations as given in [Nak75].



Dolbeault: horizontal arrows | Bott-Chern: ingoing arrows | Aeppli: outgoing arrows

FIGURE 7. Cohomologies of the Iwasawa manifold.

So, the idea here is that Dolbeault cohomology does not care horizontal arrows, conjugate Dolbeault cohomology does not care vertical arrows, Bott-Chern cohomology counts possibly incoming corners, Aeppli cohomology counts possibly outgoing corners, with the exception, in any case, of squares. Hence, just by combinatorial arguments, one recognizes that the sum of the dimension of the Bott-Chern and Aeppli cohomologies is greater or equal than the sum of the dimension of Dolbeault and conjugate Dolbeault cohomologies, which is greater or equal than twice the Betti number by the Frölicher spectral sequence. Moreover, both equalities hold if and only if the double complex is direct sum of squares and dots. That is, if the manifold satisfies the $\partial\overline{\partial}$-Lemma. A precise proof uses the Varouchas exact sequence[Var86]:

$$0 \to \frac{\operatorname{im}\overline{\partial} \cap \operatorname{im}\partial}{\operatorname{im}\partial\overline{\partial}} \to \frac{\ker\overline{\partial} \cap \operatorname{im}\partial}{\operatorname{im}\partial\overline{\partial}} \to H^{\bullet,\bullet}_{\overline{\partial}}(X) \to H^{\bullet,\bullet}_{A}(X) \to \frac{\ker\partial\overline{\partial}}{\ker\overline{\partial} + \operatorname{im}\partial} \to 0$$

and its dual, where $\overline{\partial}\colon \left(\frac{\ker\partial\overline{\partial}}{\ker\overline{\partial}+\operatorname{im}\partial}\right)^{p,q} \stackrel{\simeq}{\to} \left(\frac{\operatorname{im}\overline{\partial}\cap\ker\partial}{\operatorname{im}\partial\overline{\partial}}\right)^{p,q+1}$. □

Remark II.5.4 (Andrei Teleman [Tel06]). As for compact complex surfaces, the degrees $\Delta^k := h^k_{BC} + h^k_A - 2\, b_k$ are topological invariants. More precisely, $\Delta^1 = 0$ and $\Delta^2 \in \{0, 2\}$ according to the complex surface admitting Kähler metrics (by using also the Lamari and the Buchdahl criterion). (Note that, while Theorem II.5.3 gives an inequality of algebraic type, the result by Teleman founds on analytic properties: the ellipticity of the differential operator $\mathcal{D}(f) := dd^c f \wedge \omega^{n-1}$ on functions, with index 0 and 1-dimensional kernel, where ω is the associated $(1,1)$-form to a metric and n is the complex dimension.)

Remark II.5.5. Notice that we do not have a topological upper bound: this is because of the even-length zigzags, which contribute to the Dolbeault cohomology but not to the de Rham cohomology. On the other side, we have an upper bound for the Bott-Chern cohomology in terms of the Dolbeault cohomology [AT17]:

$$h^k_{BC} \leq (n+1)\left(h^k_{\overline{\partial}} + h^{k-1}_{\overline{\partial}}\right).$$

This is because a contribution to Aeppli cohomology arises from zigzags of positive length $\ell + 1$. Any such zigzag, when placed between total degrees k and $k+1$, creates exactly two non trivial classes in either Dolbeault or conjugate Dolbeault cohomology at either degree k or degree $k+1$, and at most $\lfloor \ell/2 \rfloor + 1$ classes in Aeppli cohomology at degree k. (In particular, $\lfloor \ell/2 \rfloor + 1 \leq \min\{k+1, (2n-k)+1\} \leq n+1$.) Use then the Schweitzer duality and the Serre duality.

We conclude this section with a remark on conjectured results towards the Hopf problem:

Remark II.5.6 (Sullivan's "scaling exponent" conjecture). We expect that $\sum_k b_k(X) \geq 3$ for any compact complex manifold X of dimension $\dim X = n \geq 3$. See also [AM18]. We also expect that $\sum_{p,q} h^{p,q}_{BC}(X) \geq n+1$.

II.6. The $\partial\overline{\partial}$-Lemma property for compact Kähler manifolds

II.6.1. **Kähler manifolds.** Let us consider a complex manifold endowed with a *Hermitian metric*, namely, a Riemannian metric g such that the complex structure J is an isometry: $g = g(J\_, J\_)$. Then $\omega := g(J\_, \_)$ is the associated $(1,1)$-form. We will confuse g, ω, and the associated Hermitian structure $h = g - \sqrt{-1}\omega$ when talking about a Hermitian metric on X.

A very special case is when g *osculates at the second order the Euclidean metric* at any point (the higher orders depending on the curvature): for every point $x \in X$, there exists a holomorphic coordinate chart $\left(U,\ \left\{z^j\right\}_{j\in\{1,\ldots,n\}}\right)$, with $x \in U$, such that

$$g = \sum_{\alpha,\beta=1}^{n} \left(\delta_{\alpha\beta} + \mathrm{o}(z)\right)\, dz^\alpha \odot d\bar{z}^\beta \qquad \text{at } x,$$

(see *e.g.* [GH94, pages 107–108], [Huy05, Proposition 1.3.12], [Mor07, Theorem 11.6].

This is equivalent to asking for the associated $(1,1)$-form ω to be closed:

$$d\omega = 0,$$

that is, ω is a *symplectic form* (*i.e.* a non-degenerate closed form) compatible with J (*i.e.* of type $(1,1)$). In this case, we say that ω is a *Kähler metric*.

The notion of Kähler manifold has been studied for the first time by J. A. Schouten and D. van Dantzig [SvD30], see also [Sch29], and by E. Kähler [Käh33], and the terminology has been fixed by A. Weil [Wei58]. On a Kähler manifold, we have then three structures,

- a complex structure J,
- a symplectic structure ω,
- a Hermitian structure g,

related each other by the condition

$$g = \omega(\_,\, J\_).$$

This can be read as the Kähler geometry being pointwise (*i.e.* forgetting integrability even at first order) governed by the structure group

$$\begin{aligned}\mathrm{U}(n) &= \mathrm{GL}(n;\mathbb{C}) \cap \mathrm{Sp}(2n;\mathbb{R}) \cap \mathrm{O}(2n;\mathbb{R}) \\ &= \mathrm{GL}(n;\mathbb{C}) \cap \mathrm{Sp}(2n;\mathbb{R}) = \mathrm{GL}(n;\mathbb{C}) \cap \mathrm{O}(2n;\mathbb{R}) = \mathrm{Sp}(2n;\mathbb{R}) \cap \mathrm{O}(2n;\mathbb{R}).\end{aligned}$$

Remark II.6.1. Note that, on a manifold X endowed with a symplectic form ω, there is always a (possibly non-integrable) almost-complex structure J such that $g := \omega(\_, J\_)$ is a Hermitian metric associated to ω, see *e.g.* [CdS01, Corollary 12.7]. (The space of such almost-complex structures is contractible, see *e.g.* [AL94, Corollary II.1.1.7], [CdS01, Proposition 13.1].)

Several equivalent definitions can be given for the compatibility. For example, on a complex manifold X with a Hermitian metric g, one can prove the formulas

$$3d\omega(x,y,z) = \langle(\nabla_x J)y, z\rangle + \langle(\nabla_y J)z, x\rangle + \langle(\nabla_z J)x, y\rangle,$$

respectively,

$$3d\omega(x,y,z) - 3d\omega(x,Jy,Jz) = 2\langle(\nabla_x J)y, z\rangle + \langle x, \mathrm{Nij}_J(y,Jz)\rangle,$$

whence it follows that the following are equivalent:

- $\nabla J = 0$;
- $d\omega = 0$.

In particular:

Corollary II.6.2. *Let X be a complex manifold endowed with a Hermitian structure g. Then g is Kähler if and only if the Levi-Civita connection and the Chern connection coincide.*

By the Poincaré Lemma, locally any Kähler ω can be written as

$$\omega \stackrel{\text{loc}}{=} \sqrt{-1}\partial\overline{\partial}g,$$

for a smooth function g.

We notice that, by the Stokes theorem, if ω is a Kähler metric, then necessarily ω^k is closed not exact, for any k; in particular, $b_{2k}(X) > 0$. Moreover, if $j\colon N \to X$ is a complex submanifold of complex dimension k (namely, N is locally defined by $n-k$ holomorphic equations with $\mathbb{C}$-linear independent differentials), then N is not a boundary.

For example, It follows that the Hopf surface $S^1 \times S^3$ admits no Kähler structure, because $b_2(X) = 0$. And the *Hironaka example* [Hir62], see also [Har77], admits no Kähler structure, because it contains a complex submanifolds being a boundary.

II.6.2. **Projective manifolds.** The Kähler case is particularly interesting for at least two reasons. First, every such a metric always exists on a projective manifold, namely, Kähler manifolds can be intended as a (transcendental) generalization of projective (algebraic) manifolds. Indeed, the so-called *Fubini-Study metric* [Fub04, Stu05] on $\mathbb{CP}^n$ is a Kähler metric. It is induced by the fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \to \mathbb{CP}^n$; more precisely, by using the homogeneous coordinates $[z_0 : \cdots : z_n]$, one has that the associated $(1,1)$-form ω_{FS} to the Fubini and Study metric is

$$\omega_{\mathrm{FS}} = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\left(\sum_{\ell=0}^{n}|z^\ell|^2\right).$$

By restrictions along analytic submanifolds, it follows that complex projective (algebraic) manifolds are Kähler manifolds. On the other side:

Theorem II.6.3 (Kodaira embedding [Kod54, Theorem 4])**.** *Let X be a compact complex manifold admitting Kähler metrics. The following are equivalent:*

- *there exists a Kähler metric ω such that*
$$[\omega] \in H^2_{dR}(X;\mathbb{R}) \cap \operatorname{im}\left(H^2(X;\mathbb{Z}) \to H^2_{dR}(X;\mathbb{R})\right);$$
- *there exists a positive line bundle, (namely, a holomorphic line bundle endowed with a Hermitian metric with positive curvature; equivalently, its first Chern class is represented by a positive form;)*
- *there exists an ample line bundle, (namely, for some tensor power $L^{\otimes k}$, for some $k \geq 1$, a basis for $H^0(X;L^{\otimes k})$ provides an embedding Φ of X into $\mathbb{CP}^{N_k}$ where $N_k := \dim H^0(X;L^{\otimes k}) - 1$); moreover, $L^{\otimes k} = \Phi^*\mathcal{O}_{\mathbb{CP}^N}(1)$, where $\mathcal{O}_{\mathbb{CP}^N}(1)$ is the dual of the tautological bundle $\mathcal{O}_{\mathbb{CP}^N}(-1)$;*
- *there exists a complex-analytic embedding of X into a complex projective space $\mathbb{CP}^N$.*

II.6.3. **Hodge theory for Kähler manifolds.** The second reason is that, by the very definition, every local property just depending on the metric and its first derivatives is true on a Kähler manifold if and only if it is true on the Euclidean space $\mathbb{C}^n$. In particular, this holds for the *Kähler identities*, that interrelate the action of the operators $L := \omega \wedge \_$, $\Lambda := -\iota_{\omega^{-1}}$, $H = [L, \Lambda]$ of the symplectic structures, of the operators d, $d^* = - * d *$ of the Riemannian structures, and of the operators $\overline{\partial}$, ∂ of the complex structure. We can prove it by local calculation on $\mathbb{C}^n$, or by exploting the $\mathfrak{sl}(2)$-representation of the symplectic structures, as we do (see *e.g.* [Huy05, Proposition 3.1.12]); see also [Hod35, Hod89]; see also [Dem86, Theorem 1.1, Theorem 2.12], and [Gri66], [Dem12, §VI.6.2], for commutation relations on arbitrary Hermitian manifolds are provided).

Theorem II.6.4 (Kähler identities, [Wei58, Théorème II.1, Théorème II.2, Corollaire II.1])**.** *Let X be a compact Kähler manifold. Consider the differential operators ∂ and $\overline{\partial}$ associated to the complex structure, the symplectic operators L and Λ associated to the symplectic structure, and the Hodge-$*$-operator associated to the Hermitian metric. Then, these operators are related as follows:*

(i) $[\overline{\partial},\, L] = [\partial,\, L] = 0$ *and* $[\Lambda,\, \overline{\partial}^*] = [\Lambda,\, \partial^*] = 0$*;*
(ii) $[\overline{\partial}^*,\, L] = \sqrt{-1}\,\partial$ *and* $[\partial^*,\, L] = -\sqrt{-1}\,\overline{\partial}$*, and* $[\Lambda,\, \overline{\partial}] = -\sqrt{-1}\,\partial^*$ *and* $[\Lambda,\, \partial] = \sqrt{-1}\,\overline{\partial}^*$.

Therefore, the 2*nd order self-adjoint elliptic differential operators* $\square := [\partial,\, \partial^*]$, $\overline{\square} := \left[\overline{\partial},\, \overline{\partial}^*\right]$, $\Delta := [d,\, d^*]$, $\tilde{\Delta}_{BC}$ *and* $\tilde{\Delta}_A$*, are related as follows:*

(iii) $\square = \overline{\square} = \frac{1}{2}\Delta$*, and* Δ *commutes with* $*,\ \partial,\ \overline{\partial},\ \partial^*,\ \overline{\partial}^*,\ L,\ \Lambda$*;*
(iv) $\tilde{\Delta}_{BC} = \overline{\square}^2 + \partial^*\partial + \overline{\partial}^*\overline{\partial}$*,* [KS60, Proposition 6], [Sch07, Proposition 2.4]*;*
(v) $\tilde{\Delta}_A = \overline{\square}^2 + \partial\partial^* + \overline{\partial}\overline{\partial}^*$.

Sketch of the proof. The previous identities can be proven either using the $\mathfrak{sl}\,(2;\mathbb{C})$-representation $\langle L,\, \Lambda,\, H\rangle \to \mathrm{End}^\bullet\left(\wedge^\bullet X \otimes \mathbb{C}\right)$, or reducing to prove the corresponding identities on $\mathbb{C}^n$ with the standard Kähler structure (which are known as Y. Akizuki and S. Nakano's identities, [AN54, §3]) and hence using that every Kähler metric osculates to order 2 the standard Hermitian metric on $\mathbb{C}^n$.

We briefly recall the $\mathfrak{sl}(2;\mathbb{R})$-representation induces by a symplectic structure on the space of forms. Let X be a manifold endowed with a symplectic structure ω. Denote by the map $\iota_\xi\colon \wedge^\bullet X \to \wedge^{\bullet-2}X$ the interior product with $\xi \in \wedge^2(TX)$, and by the map $\pi_{\wedge^k X}\colon \wedge^\bullet X \to \wedge^k X$ the natural projection onto $\wedge^k X$. Finally, the canonical Poisson bi-vector is $\omega^{-1}(\_,\_) := \omega(I^{-1}\_, I^{-1}\_)$, where $I\colon TX \to T^*X$ is the natural isomorphism induced by ω. In a Darboux coordinate chart $\left(U,\, \{x^j\}_{j\in\{1,\dots,2n\}}\right)$, we have $\omega = \sum dx^{2j-1} \wedge dx^{2j}$ and $\omega^{-1} \stackrel{\text{loc}}{=} \sum_{j=1}^n \frac{\partial}{\partial x^{2j-1}} \wedge \frac{\partial}{\partial x^{2j}}$. Consider the following operators:

$$\begin{aligned} L &:= \omega \wedge \_\colon \wedge^\bullet X \to \wedge^{\bullet+2}X, \\ \Lambda &:= -\iota_{\omega^{-1}}\colon \wedge^\bullet X \to \wedge^{\bullet-2}X, \\ H &:= \sum_k (n-k)\pi_{\wedge^k X}\colon \wedge^\bullet X \to \wedge^\bullet X. \end{aligned}$$

Then we have the $\mathfrak{sl}(2;\mathbb{R})$*-representation* [Yan96, Corollary 1.6] of $\mathfrak{sl}(2;\mathbb{R}) \simeq \langle L, \Lambda, H\rangle \to \mathrm{End}^\bullet\left(\wedge^\bullet X\right)$, namely

$$[L,H] = 2\,L, \qquad [\Lambda, H] = -2\Lambda, \qquad [L,\Lambda] = H.$$

This $\mathfrak{sl}(2;\mathbb{R})$-representation have *finite H-spectrum* (namely, we can decompose $\wedge^\bullet X$ into the direct sum of eigen-spaces of H and H has only finitely-many distinct eigen-values, [Yan96, Definition 2.2]). Therefore it induces a decomposition of the space of the differential forms, the *Lefschetz decomposition*:

$$\wedge^\bullet X = \bigoplus_{r\in\mathbb{N}} L^r\, \mathrm{Prim}^{\bullet-2r} X,$$

where the space of *primitive forms* is

$$\mathrm{Prim}^\bullet X := \ker\Lambda = \ker L^{n-k+1}\lfloor_{\wedge^k X}.$$

Without relying on results on $\mathfrak{sl}(2;\mathbb{R})$-representations, we provide a concrete proof of this last result. Denote by $2n$ the dimension of X. Recall that $[L,\Lambda]\lfloor_{\wedge^k X} = (n-k)\,\mathrm{id}$, whence by induction $[L^\ell,\Lambda]\lfloor_{\wedge^k X} = -j(k-n+j-1)L^{j-1}$. For induction on $j \leq n-k$, we show that $L^{j-1}\lfloor_{\wedge^k X}$ is injective. It follows also that $L^k\lfloor_{\wedge^{n-k}X}\colon \wedge^{n-k}X \to \wedge^{n+k}X$ is an isomorphism. This says also that $\ker\Lambda\lfloor_{\wedge^k X} = \ker L^{n-k+1}$. By induction, we can now prove uniqueness and existence of the Lefschetz decomposition.

We continue the proof of the Kähler identities. It suffices to prove that $[\Lambda, d] = -(d^c)^*$ on forms of type $L^j\beta$ with $\beta\in\mathrm{Prim}^k X$ primitive, where $d^c = J^{-1}dJ = -\sqrt{-1}\,(\partial-\overline{\partial})$ with J the complex structure, and its adjoint with respect to the metric structure is $(d^c)^* = -*d^c*$. Denote by n the complex dimension of X. By using that $L^j\lfloor_{\wedge^k X}$ is injective for $j\leq n-k$, one shows that $d\alpha = \beta_0 + L\beta_1$ with β_0, β_1 primitive forms. Use also

$$[L^j,\Lambda]\lfloor_{\wedge^k X} = -j\,(k-n+j.1)L^{j-1}.$$

It yields $[\Lambda,d](L^j\alpha) = -jL^{j-1}\alpha_0 - (k-n+j-1)L^j\alpha_1$. On the other hand, to compute $(d^c)^*(L^j\alpha)$, we use the formula

$$*L^j\lfloor_{\mathrm{Prim}^k X} = (-1)^{k(k+1)/2}\,\frac{j!}{(n-k-j)!}\,L^{n-k-j}J.$$

A straightforward computation yields $-(d^c)^*L^j\alpha = -jL^{j-1}\alpha_0 - (k-n+j-1)L^j\alpha_1$, completing the proof.

The equality between the Laplacians follows by first showing that $\partial\overline{\partial}^* + \overline{\partial}^*\partial = 0$, and then showing that $\square = \overline{\square}$, and finally showing that $\Delta = 2\square$, thanks to the previous identities.

In the same way, we get the commutation $[\Delta, L] = 0$. $\square$

As a consequence of the last statement, with respect to a Kähler metric, the space of harmonic forms with respect to the different Laplacians actually coincide, and we can move from different Hodge decompositions. Thanks to the Hodge theory for the several Laplacians, we get a decomposition in cohomology, which *a priori* depends on the chosen Kähler metric (actually, it is not, and this will be the contents of the $\partial\overline{\partial}$-Lemma): see, *e.g.*, [Huy05, Corollary 3.2.12], respectively [Huy05, Proposition 3.2.13].

Theorem II.6.5 (Hodge decomposition theorem, [Wei58, Théorème IV.3])**.** *Let X be a compact complex manifold endowed with a Kähler structure. Then there exist a decomposition*

$$H^{\bullet}_{dR}(X;\mathbb{C}) \simeq \bigoplus_{p+q=\bullet} H^{p,q}_{\overline{\partial}}(X) ,$$

and isomorphisms

$$H^{p,q}_{\overline{\partial}}(X) \simeq \overline{H^{q,p}_{\overline{\partial}}(X)} .$$

Moreover, the above decomposition is orthogonal with respect to the (possibly non positive-definite) Hermitian form on $H^k_{dR}(X;\mathbb{C})$ defined as

$$H_k(\alpha,\beta) := \sqrt{-1}^k \int \omega^{n-k} \wedge \alpha \wedge \bar{\beta}.$$

Theorem II.6.6 (Lefschetz decomposition theorem, [Wei58, Théorème IV.5])**.** *Let X be a compact complex manifold, of complex dimension n, endowed with a Kähler structure. Then there exist a decomposition*

$$H^{\bullet}_{dR}(X;\mathbb{C}) = \bigoplus_{r\in\mathbb{N}} L^r \mathrm{Prim} H^{\bullet-2r}(X),$$

where $\mathrm{Prim}H^{\bullet-2r}(X) := \left(\ker\left(\Lambda\colon H^{\bullet-2r}_{dR}(X;\mathbb{C}) \to H^{\bullet-2r-2}_{dR}(X;\mathbb{C})\right)\right)$, *and isomorphisms*

$$L^k\colon H^{n-k}_{dR}(X;\mathbb{C}) \stackrel{\simeq}{\to} H^{n+k}_{dR}(X;\mathbb{C}) .$$

Moreover, the above decomposition is orthogonal with respect to the (possibly non positive-definite) Hermitian form on $H^k_{dR}(X;\mathbb{C})$ defined as

$$H_k(\alpha,\beta) := \sqrt{-1}^k \int \omega^{n-k} \wedge \alpha \wedge \beta.$$

The two decompositions are related by the following:

Theorem II.6.7 (Hodge index theorem)**.** *Let X be a compact complex manifold, of complex dimension n, endowed with a Kähler structure. On $H^k_{dR}(X;\mathbb{C})$, consider the (possibly non positive-definite) Hermitian form on $H^k_{dR}(X;\mathbb{C})$ defined as*

$$H_k(\alpha,\beta) := \sqrt{-1}^k \int \omega^{n-k} \wedge \alpha \wedge \beta.$$

Then, on $H^{p,q}_{\overline{\partial}}(X) \cap \mathrm{Prim}H^k X$, the Hermitian form

$$(-1)^{k(k-1)/2} \sqrt{-1}^{p-q-k} H_k$$

is positive-definite.

In particular, when n is even, the (symmetric) intersection form on $H^n(X)$,

$$Q(\alpha,\beta) = \int_X \alpha \wedge \beta,$$

has signature

$$\sum_{a,b} (-1)^a h^{a,b}.$$

A useful tool here is the formula

$$*\lfloor_{\wedge^{p,q}X\cap\mathrm{Prim}^kX} = (-1)^{k(k-1)/2}\,\sqrt{-1}^{p-q}\,\frac{1}{(n-k)!}\,L^{n-k}.$$

As we announced above, the Hodge decomposition actually does not depend on the chosen Kähler metric. In fact:

Theorem II.6.8 (Deligne-Griffiths-Morgan-Sullivan [DGMS75]). *Compact Kähler manifolds satisfy the $\partial\overline{\partial}$-Lemma.*

Proof. Once fixed a Hermitian metric, the Kähler identities ($[\omega\wedge\_, d]=0$ and $[\omega\wedge\_, d^*]=d^c$) yield the comparison for the Laplacian:

$$\tilde{\Delta}_{BC} = \overline{\square}^2 + \partial^*\partial + \overline{\partial}^*\overline{\partial},$$

which yields an unnatural isomorphism between the cohomologies.

More precisely, argue as follows by using the Hodge decomposition once fixed a Hermitian metric. Let $\alpha\in\wedge^{p,q}X$ be a ∂-closed $\overline{\partial}$-closed d-exact form on X. In particular, by the Hodge decomposition theorem for the de Rham cohomology, the formα is orthogonal to the space of Δ-harmonic forms. Note that, by the Kähler identities, the space of Δ-harmonic forms coincide with the space of $\square$-harmonic forms and with the space of $\overline{\square}$-harmonic forms. Since α is ∂-closed and orthogonal to the space of $\square$-harmonic forms, the conjugate version of the Hodge decomposition theorem for the Dolbeault cohomology yields $\alpha=\partial\gamma$ for some $\gamma\in\wedge^{p-1,q}X$. By applying the Hodge decomposition theorem for the Dolbeault cohomology, to the form γ, one gets a $\overline{\square}$-harmonic form h_γ, a form $\beta\in\wedge^{p-1,q-1}X$, and a form $\eta\in\wedge^{p-1,q+1}X$ such that $\gamma=h_\gamma+\overline{\partial}\beta+\overline{\partial}^*\eta$. By the Kähler identities, one has $\left[\partial,\overline{\partial}^*\right]=0$ and that h_γ is also $\square$-harmonic. Hence $\alpha=\partial\gamma=\partial\overline{\partial}\beta-\overline{\partial}^*\partial\eta$. It suffices to prove that $\overline{\partial}^*\partial\eta=0$. Indeed, since α is $\overline{\partial}$-closed, one has $\overline{\partial}\overline{\partial}^*\partial\eta=0$, and hence $\left\|\overline{\partial}^*\partial\eta\right\|^2=\left\langle\overline{\partial}^*\partial\eta,\,\overline{\partial}^*\partial\eta\right\rangle=\left\langle\overline{\partial}\overline{\partial}^*\partial\eta,\,\partial\eta\right\rangle=0$. Hence $\alpha=\partial\overline{\partial}\beta$ is $\partial\overline{\partial}$-exact. $\square$

The $\partial\overline{\partial}$-Lemma property is extremely useful in Kähler geometry, for example, in reducing the study of Kähler metrics belonging to a fixed Kähler class to their potential functions. Let us look closely to the following problem, known as the *Calabi-Yau problem*. Given any Kähler metric ω, its Ricci form $\rho(\omega)$ is a d-closed $(1,1)$-form belonging to the first Chern class $c_1(X):=c_1(K_X^{-1})$, which just depends on the complex structure. We ask whether any representative in $c_1(X)$ is actually the Ricci form of some Kähler metric in the same cohomology class as ω. By the $\partial\overline{\partial}$-Lemma, we can confuse de Rham cohomology classes and Bott-Chern cohomology classes. Namely, given $\Psi=\rho(\omega)+\sqrt{-1}\partial\overline{\partial}\psi\in c_1(X)$, we are looking for $\tilde{\omega}=\omega+\sqrt{-1}\partial\overline{\partial}\varphi>0$ such that $\rho(\tilde{\omega})=\Psi$. By the local expression

$$\mathrm{Ric}^{Ch}\stackrel{\text{loc}}{=}\sqrt{-1}\overline{\partial}\partial\log\omega^n,$$

this reduces to

$$\sqrt{-1}\partial\overline{\partial}\log(\omega+\sqrt{-1}\partial\overline{\partial}\varphi)^n=\sqrt{-1}\partial\overline{\partial}\log\omega^n+\sqrt{-1}\partial\overline{\partial}\psi,$$

whence

$$(\omega+\sqrt{-1}\partial\overline{\partial}\varphi)^n=\exp(\psi+c)\omega^n,$$

where c is a constant. So we have reduced the problem to a scalar equation in ϕ, that can be solve by using the continuity method and the Yau *a priori* estimates [Yau78].

II.6.4. **Hodge conjecture for projective manifolds.** Summarizing the contents of the previous section, on X compact Kähler of complex dimension n, we have

$$H^k(X;\mathbb{C})=H^k(X;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C}=H^k(X;\mathbb{Z})/\mathrm{tor}\otimes_{\mathbb{Z}}\mathbb{C}$$

and the Hodge decomposition

$$H^k(X;\mathbb{C})=\bigoplus_{p+q=k}H^{p,q}(X)\qquad\text{with}\qquad H^{p,q}(X)=\overline{H^{q,p}(X)}.$$

In particular, we say that $H^k(X;\mathbb{C})$ admits an *integral Hodge structure of weight* k, namely, we have a torsion-free f.g. Abelian group $V_{\mathbb{Z}}=H^k(X;\mathbb{Z})/\mathrm{tor}$ and a decomposition $H^k(X;\mathbb{C})=V_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{C}=\bigoplus_{p+q=k}V^{p,q}$ with $V^{p,q}=\overline{V^{q,p}}$.

Moreover, the Kähler form ω induces a Hermitian form

$$H(\alpha,\beta)=\sqrt{-1}^k\int_X\omega^{n-k}\wedge\alpha\wedge\bar{\beta}$$

on $H^k(X;\mathbb{C})$ with respect to which the Hodge decomposition is an orthogonal decomposition, and we have that $(-1)^{k(k-1)/2}\sqrt{-1}^{p-q-k}H$ is positive-definite on $\mathrm{Prim}\,H^{p,q}(X)$. And the Lefschetz decomposition and the above intersection form are well-defined over $H^k(X;\mathbb{Z})$ when $[\omega]\in H^2(X;\mathbb{Z})\cap H^{1,1}(X)$, say it is a *Hodge class*; by the Kodaira embedding, this happens if and only if X admits a holomorphic embedding in $\mathbb{CP}^N$.

Remark II.6.9. Note that a Hodge decomposition determines a Hodge decreasing filtration $F^pV_{\mathbb{C}}=\bigoplus_{r\geq p}V^{r,k-r}$ such that $V_{\mathbb{C}}=F^pV_{\mathbb{C}}\oplus\overline{F^{k-p+1}V_{\mathbb{C}}}$. Conversely, such a filtration determines a Hodge decomposition by setting $V^{p,q}=F^pV_{\mathbb{C}}\cap\overline{F^qV_{\mathbb{C}}}$.

We want to understand better the geometric meaning of integral classes of type (k,k).

Consider an *analytic subset* of a complex manifold X, namely, a closed subset $Z\subset X$ that is locally expressed as zero set of holomorphic functions. By the Implicit Function Theorem, any complex submanifold is an analytic subset; on the other side, any analytic subset Z has a dense open smooth *regular* part Z_{smooth} and a nowhere-dense *singular* analytic subset Z_{sing}. The analytic subset Z is called *irreducible* if Z_{smooth} is connected, and in this case we can define its *dimension* as the dimension of the smooth part. By the Weierstrass Preparation Theorem, we get that an analytic subset is stratified by closed analytic subsets Z_k of strictly decreasing dimension such that $Z_k\setminus Z_{k+1}$ is a closed submanifold of $X\setminus Z_{k+1}$.

By the exact sequence of relative cohomology, if $Y\subset X$ is a complex sumanifold of codimension k, then the map $H^j(X;\mathbb{Z})\to H^j(X\setminus Y;\mathbb{Z})$ is an isomorphism for $j<2k$. This yields that, for an analytic subset Z of codimension r, the class $[Z]\in H^{2r}(X;\mathbb{Z})$ is well-defined as the Poincaré dual of its smooth part $[Z_{\mathrm{smooth}}]$.

More concretely, $[Z]$ is the class $\varphi \mapsto \int_Z \varphi := \int_{Z_{\text{smooth}}} \varphi$, where the integral is well-defined because locally Z_{smooth} is a d-sheeted ramified covering on a disc; for the same reason, the Stokes theorem still holds. This makes clear that, if X admits a Hodge structure (*e.g.* X is projective), then $[Z] \in H^{2r}(X;\mathbb{Z}) \cap H^{r,r}(X)$.

We define the *analytic cycles* of codimension r as the group $\mathcal{Z}^r(X)$ generated by analytic subsets of codimension r. By the previous argument, we have the *cycle class map*

$$\mathcal{Z}^r(X) \to H^{2r}(X;\mathbb{Z}).$$

Its image $H^{2r}(X;\mathbb{Z})_{\text{an}}$ is called the group of *analytic classes*. On a Kähler manifold, analytic classes are *Hodge classes* of degree $2r$, namely, integer classes of type (r,r), *i.e.* in $H^{2r}(X;\mathbb{Z})_{\text{an}} \subseteq H^{r,r}(X)$. In the projective algebraic case, we speak of *algebraic classes*, say $H^{2r}(X;\mathbb{Z})_{\text{alg}}$.

Conjecture II.6.10 (rational Hodge conjecture)**.** *Let X be a projective manifold. Then any Hodge class of type degree $2k$ is represented by an integer multiple of the class of an algebraic cycle, that is,*

$$H^{2k}(X;\mathbb{Q})_{\text{alg}} = H^{2k}(X;\mathbb{Q}) \cap H^{k,k}(X).$$

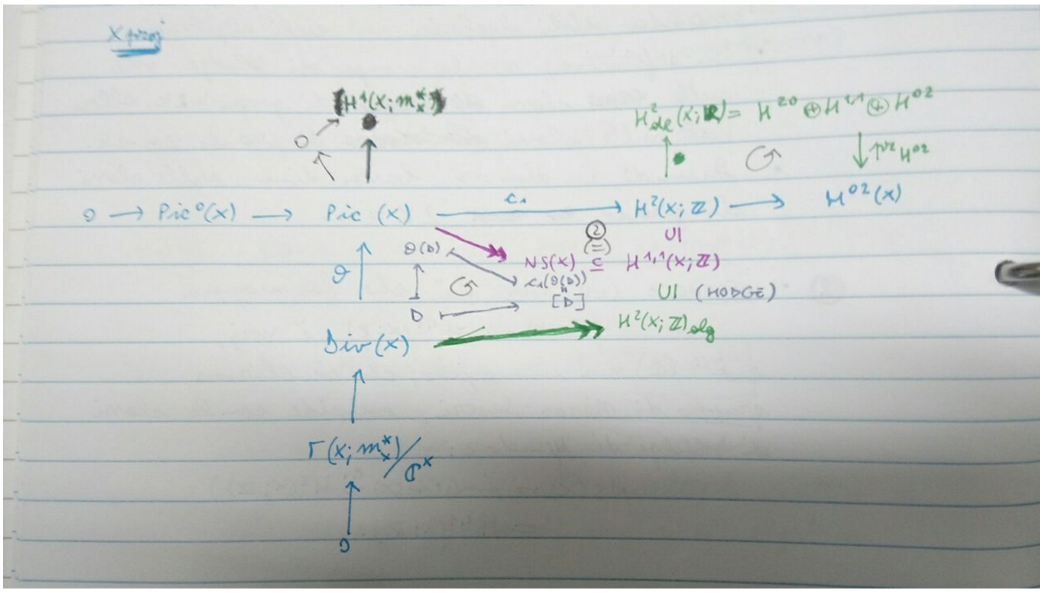
Remark II.6.11. Originally, the formulation of the Hodge conjecture [Hod89, Hod52] was "more ambitious". But Grothendieck [Gro69] pointed out that it was "false for trivial reasons". In the integral setting, the conjecture is false because of torsion classes [AH62], and also in the torsion-free case by Kollár [BCC92]. In the Kähler case, Voisin [Voi02b] provided a counter-example. As far as now, the conjecture is known to be true: for projective threefolds (as consequence of the Leftschetz $(1,1)$ theorem see below); for uniruled projective manifolds of dimension 4; when the cohomology is generated in degree 2, for example, for general Abelian varieties [Mat58], for products of elliptic curves [Tat65], for simple Abelian varieties of prime dimension [Tan82].

The Hodge conjecture is known to be true at the level of divisors, namely, codimension 1 analytic cycles; as a corollary, thanks to the Hard Lefschetz Theorem, it holds true for threefolds.

Theorem II.6.12 (Lefschetz $(1,1)$ theorem)**.** *On a projective manifold, it holds*

$$H^2(X;\mathbb{Z})_{\text{alg}} = H^2(X;\mathbb{Z}) \cap H^{1,1}(X).$$

The steps in the proof are summarized in the following.



Here, the first Chern class is defined taking the exponential sequence

$$0 \to \mathbb{Z} \stackrel{2\pi\sqrt{-1}}{\to} \mathcal{O}_X \stackrel{\exp}{\to} \mathcal{O}_X^\times \to 0,$$

and taking the associated long exact sequence in cohomology:

$$\begin{array}{ccccc} H^1(X;\mathbb{Z}) & \longrightarrow & H^1(X;\mathcal{O}_X) & \longrightarrow & H^1(X;\mathcal{O}_X^\times) \\ & & & \swarrow & \\ H^2(X;\mathbb{Z}) & \longrightarrow & H^2(X;\mathcal{O}_X). & & \end{array}$$

Here, the last map can be factorized as $H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{C}) \to H^{0,2}_{\overline{\partial}}(X)$ where the second map is induced by the projection. The group $H^1(X;\mathcal{O}_X^\times)$ parametrized the line bundles over X: it is called the *Picard group* of X. Finally, the map

$$c_1 \colon H^1(X;\mathcal{O}_X^\times) \to H^2(X;\mathbb{Z})$$

associates to any line bundle its first *Chern class*, which is an integral class that sits in $H^{1,1}(X)$. In particular, $\mathrm{Pic}^0(X) := \ker c_1 \simeq H^1(X;\mathcal{O}_X)/H^1(X;\mathbb{Z})$ is a torus that parametrizes line bundles with zero first Chern class, called the *Picard torus*; when X is projective, then the Picard torus is an Abelian variety.

Remark II.6.13. For a complex manifold X, the determinant of the holomorphic cotangent bundle determines a line bundle, the *canonical bundle*. The first *Chern class* of the manifold is then defined as the first Chern class of its canonical bundle. A metric on the manifold determines a metric on the canonical bundle by the volume. By the previous result, we can prescribe the curvature in $c_1(X)$ of the metric on the canonical bundle. On the other side, to realize such a metric on the canonical bundle as the determinant of a metric h on X, we need to prescribe the volume of h. This is the Calabi conjecture, solved by Yau [Yau78].

We also recall that, once fixed a Hermitian structure on a line bundle, there exists unique the *Chern connection* that preserves the Hermitian structure and whose $(0,1)$-components on sections of L is given by the holomorphic operator of L. Its curvature is represented by a closed $(1,1)$-form, then it induces a cohomology class in $H^{1,1}(X)$. We have the following.

Theorem II.6.14. *On a line bundle with a Hermitian structure, the class of the curvature of the Chern connection is $c_1(L)$. Conversely, on a line bundle, for any representative $\omega \in c_1(L)$, there exists a Hermitian structure on L whose curvature is ω.*

The vertical map follows by the long sequence associated to the exact sequence of sheaves

$$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \frac{\mathcal{M}^*}{\mathcal{O}^*} \to 0.$$

In particular, there appears $H^1(X;\mathcal{O}^*) = \mathrm{Pic}(X)$ and $H^0(X;\frac{\mathcal{M}^*}{\mathcal{O}^*}) = \mathrm{Div}(X)$ the group of divisors of X. The map $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ is shown to be surjective.

We first notice that c_1 is surjective onto the space of Hodge classes of degree 2, namely, the right-hand side in the statement of the Hodge conjecture equals the *Néron-Severi group*

$$\mathrm{im}\left(c_1 \colon \mathrm{Pic}(X) \to H^2(X;\mathbb{Z})\right).$$

The statement follows by the Lelong theorem, stating that $c_1(\mathcal{O}(D)) = [D]$ for D divisor, making the diagram commutative..

Corollary II.6.15. *The Hodge conjecture holds true for projective threefolds.*

II.7. Behaviour of the $\partial\overline{\partial}$-Lemma under deformations and modifications

II.7.1. **Behaviour of the $\partial\overline{\partial}$-Lemma under deformations.** We recall that the existence of Kähler metric is stable under small deformations, namely, it is an open property in the space of deformations, thanks to a result by Kodaira and Spencer [KS60], that makes advantage of elliptic operators and Hodge theory for Kähler metrics.

More in general, the validity of the $\partial\overline{\partial}$-Lemma property is stable under small deformations. The following result is due to Wu, and different proofs appeared in the literature [Voi02a, Tom08, AT13]: we give an argument exploiting the numerical characterization in Theorem II.5.3 by [AT13].

Theorem II.7.1 (Wu [Wu06])**.** *The property of satisfying the $\partial\overline{\partial}$-Lemma is stable under small deformations of the complex structure.*

Proof. Indeed, the rhs of (4) is constant under small deformation by the Ehresmann lemma, while the lhs varies upper-semi-continuously, whence the equality remains true and Theorem II.5.3 applies. □

On the other side:

Theorem II.7.2 (Angella-Kasuya [AK17b])**.** *The property of satisfying the $\partial\overline{\partial}$-Lemma is not closed under deformations.*

Proof. Explicit computations on the Lie group $\mathbb{C} \ltimes_{\mathrm{diag}(\exp z, \exp(-z))} \mathbb{C}^2$, see Section I.4.3. □

II.7.2. **Behaviour of the $\partial\overline{\partial}$-Lemma under modifications.** The following result allow to enlarge the class of Moĭšezon manifolds [Moĭ67] and of manifolds in class $\mathcal{C}$ of Fujiki [Fuj78] in the class of compact complex manifolds satisfying the $\partial\overline{\partial}$-Lemma:

Theorem II.7.3 (Deligne-Griffiths-Morgan-Sullivan [DGMS75])**.** *Let X and Y be compact complex manifolds of the same dimension, and let $f\colon X \to Y$ be a holomorphic birational map. If X satisfies the $\partial\overline{\partial}$-Lemma, then also Y satisfies the $\partial\overline{\partial}$-Lemma.*

Proof. Indeed, one has that, if X and Y are complex manifolds of the same dimension, and $\pi\colon X \to Y$ is a proper surjective holomorphic map, then the maps

$$\pi^*\colon H^{\bullet}_{dR}(Y;\mathbb{C}) \to H^{\bullet}_{dR}(X;\mathbb{C}) \qquad \text{and} \qquad \pi^*\colon H^{\bullet,\bullet}_{\overline{\partial}}(Y) \to H^{\bullet,\bullet}_{\overline{\partial}}(X)$$

induced by $\pi\colon X \to Y$ are injective. $\square$

Conversely, we expect to prove that the $\partial\overline{\partial}$-Lemma is defined inside the localization of the category of holomorphic maps with respect to bimeromorphisms. A first partial result is given by:

Theorem II.7.4 (Stelzig [Ste18a]; see also Angella-Suwa-Tardini-Tomassini [ASTT17]; Rao-Yang-Yang [RYY17])**.** *Let X be compact complex manifold, Z be a complex submanifold. If both X and Z satisfy the $\partial\overline{\partial}$-Lemma, then also the blowup $\mathrm{Bl}_Z X$ satifies the $\partial\overline{\partial}$-Lemma.*

The following are natural questions. Is the $\partial\overline{\partial}$-Lemma property preserved by submanifolds? In the following proof [ASTT17], we will also assume Z to have a holomorphic tubular neighbourhood: does the MacPherson deformation to the normal cone work to avoid this assumption?

Note that existence of Hodge structures in cohomology is not preserved by blowups (see Voisin, Vuletescu). By using the Weak Factorization Theorem for bimeromorphic maps in the analytic category, proven by Abramovich, Karu, Matsuki, Włodarczyk [AKMW02]:

Corollary II.7.5 (Stelzig [Ste18a])**.** *For compact complex threefolds, the $\partial\overline{\partial}$-Lemma is a bimeromorphic invariant. The same happens in higher dimension, if we can prove that it is a restriction invariant.*

Sketch of the proof of Theorem II.7.4 as in [ASTT17]. The several proofs essentially found on the derivation of a formula for the Dolbeault cohomology of the blowup, and then concluding thanks with arguments of the type of the E_1-isomorphism notion in [Ste18a].

We will prove Theorem II.7.4 by sheaf-theoretic, more precisely Čech-cohomological arguments, under the assumption that the center Z admits a holomorphically contractible neighbourhood, and another technical assumption that will appear later.

Let us start by looking at the de Rham cohomology [Voi02a]. Consider the long exact sequence for pairs $(X^n, X^n \setminus Z^{n-k})$ and $(\tilde{X}^n, \tilde{X}^n \setminus E^{n-1})$, use excision on a tubular neighbourhood of Z in X, and the Thom isomorphism:

$$\begin{tikzcd}
\cdots \arrow[r] & H^{h-1}(X\setminus Z;\mathbb{C}) \arrow[r] \arrow[ddd, "\simeq"] & H^h(X, X\setminus Z;\mathbb{C}) \arrow[r] \arrow[d, "\simeq"] & H^h(X;\mathbb{C}) \arrow[r] \arrow[ddd] & H^h(X\setminus Z;\mathbb{C}) \arrow[r] \arrow[ddd, "\simeq"] & \cdots \\
& & H^{h-2k}(Z;\mathbb{C}) \arrow[ur] \arrow[d] & & & \\
& & H^{h-2}(E;\mathbb{C}) \arrow[dr] \arrow[d, "\simeq"] & & & \\
\cdots \arrow[r] & H^{h-1}(\tilde{X}\setminus E;\mathbb{C}) \arrow[r] & H^h(\tilde{X}, \tilde{X}\setminus E;\mathbb{C}) \arrow[r] & H^h(\tilde{X};\mathbb{C}) \arrow[r] & H^h(\tilde{X}\setminus E;\mathbb{C}) \arrow[r] & \cdots
\end{tikzcd}$$

If the diagram commutes, as it happens both in the Kähler [Voi02a] and in the non-Kähler case, one deduces that $H^h(\tilde{X}, \tilde{X}\setminus E;\mathbb{C})$ is the pushout of the diagram $H^{h-2}(E;\mathbb{C}) \leftarrow H^{h-2k}(Z;\mathbb{C}) \to H^h(X;\mathbb{C})$.

We would like to do the same for the Dolbeault cohomology. Consider the long exact sequence for pairs. Assume that there exists a holomorphically contractible neighbourhood U of Z in X, and use the Suwa's Thom isomorphism [Suw09]:

$$\begin{tikzcd}
\cdots \arrow[r] & H^{p,q-1}(X, X\setminus Z) \arrow[r] \arrow[ddd, "\simeq"] & H^{p,q}(X, X\setminus Z) \arrow[r] \arrow[d] & H^{p,q}(X) \arrow[r] \arrow[ddd] & H^{p,q}(X\setminus Z) \arrow[r] \arrow[ddd, "\simeq"] & \cdots \\
& & \check{H}^{p,q}_{\overline{\partial}}(U, U\setminus Z) \oplus H^{p-k,q-k}(Z) \arrow[ur] \arrow[d, "\simeq"'] & & & \\
& & \check{H}^{p,q}_{\overline{\partial}}(\tilde{U}, \tilde{U}\setminus E) \oplus H^{p-1,q-1}(E;\mathbb{C}) \arrow[dr] \arrow[d] & & & \\
\cdots \arrow[r] & H^{p,q-1}(\tilde{X}, \tilde{X}\setminus E) \arrow[r] & H^{p,q}(\tilde{X}, \tilde{X}\setminus E) \arrow[r] & H^{p,q}(\tilde{X}) \arrow[r] & H^{p,q}(\tilde{X}\setminus E) \arrow[r] & \cdots
\end{tikzcd}$$

Under another technical assumption, see [ASTT17], we can assure that the diagrams are commutative and that $H^{p,q}(\tilde{X})$ is the pushout of $H^{p-1,q-1}(E;\mathbb{C}) \leftarrow H^{p-k,q-k}(Z;\mathbb{C}) \to H^{p,q}(X)$.

By the naturality of the limit, we conclude that, if the $\partial\overline{\partial}$-Lemma holds for both Z and X (and clearly it holds for E, that is just the projectivization of the normal bundle of Z in X, and whose cohomology can be explicitly described), then it holds also for $\tilde{X}$. $\square$

II.8. Topological obstructions for the $\partial\overline{\partial}$-Lemma

A topological obstruction to the validity of $\partial\overline{\partial}$-Lemma is given by the notion of *formality* that we explain now.

Let us focus on the differential graded algebra structure on the space of forms given by the wedge product and the exterior differential, and on the de Rham cohomology. By the Leibniz rule, it induces a structure of algebra in cohomology. We look at H_{dR} as a functor inside the category **dga** of differential $\mathbb{Z}$-graded algebras:

$$H_{dR} \colon \mathbf{dga} \rightsquigarrow \mathbf{dga} \,.$$

We ask for what objects X this functor can be made "concrete", that is, when it can be realized as a composition of quasi-isomorphisms and formal inverses of quasi-isomorphisms in **dga**: e.g.,

$$\begin{tikzcd}
X \arrow[rrrrrrr, rightsquigarrow] \arrow[dr, "\text{qis}"'] & & & & & & & H_{dR}(X) \\
& C_1 & C_2 \arrow[l, "\text{qis}"] & C_3 \arrow[l, "\text{qis}"] \arrow[r, "\text{qis}"'] & \cdots \arrow[r, "\text{qis}"'] & C_{h-1} & C_h \arrow[l, "\text{qis}"] \arrow[ur, "\text{qis}"'] &
\end{tikzcd}$$

By the existence of minimal models, this corresponds to the dga of forms and the dga of de Rham cohomology sharing the same model. A compact complex manifold whose double complex of forms satisfies such a property is called *formal* in the sense of Sullivan [Sul76]. Note that the minimal model contains informations on the rational homotopy groups of the manifold [Sul76]: hence the rational homotopy type of formal manifolds is a formal consequence of their de Rham cohomology.

In fact, this can always be done in the category of A_∞-algebras (that is, strongly homotopy associative algebras), thanks to the Homotopy Transfer Principle by Kadeishvili. By [LPWZ09], the induced A_∞-structure in cohomology yields the Massey products, up to sign.

With these notations, a compact complex manifold is formal in the sense of Sullivan if there exists a system of representatives $H^\bullet$ for the cohomology such that the induced A_∞-structure on $H^\bullet$ is actually an algebra structure. (A particular case is when the chosen representatives are actually the harmonic representatives with respect to some Hermitian metric. This last situation is referred as *geometric formality* in the sense of Kotschick [Kot01].)

The "Main Theorem" in [DGMS75] is the following:

Theorem II.8.1 (Deligne-Griffiths-Morgan-Sullivan [DGMS75])**.** *Let X be a compact differentiable manifold. If X admits a complex structure such that the $\partial\overline{\partial}$-Lemma holds, then the differential graded algebra $(\wedge^\bullet X,\, d)$ is formal. In particular, all the Massey products of any order are zero.*

Proof. Indeed, if X satisfies the $\partial\overline{\partial}$-Lemma, then the inclusion $\ker d^c \to \wedge^\bullet X$ (where $d^c := J^{-1} dJ$) and the projection $\ker d^c \to \frac{\ker d^c}{\operatorname{im} d^c}$ induce the quasi-isomorphisms

$$\begin{tikzcd}
& (\ker d^c,\, d) \arrow[dl, "\text{qis}"] \arrow[dr, "\text{qis}"'] & \\
(\wedge^\bullet X,\, d) & & \left(\frac{\ker d^c}{\operatorname{im} d^c},\, 0\right)
\end{tikzcd}$$

proving that $(\wedge^\bullet X,\, d)$ is equivalent to $\left(\frac{\ker d^c}{\operatorname{im} d^c},\, 0\right)$, and hence formal. $\square$

II.8.1. **Duality for Bott-Chern cohomology.** A *Dolbeault homotopy theory* has been developed in [NT78]. Let us focus now on the algebraic structure of Bott-Chern and Aeppli cohomologies. Note that the wedge products on forms induces an algebra structure in Bott-Chern cohomology, and just a H_{BC}-module structure in Aeppli cohomology. The duality pairing given by any fixed Hermitian metric is internal in de Rham and Dolbeault cohomologies by Poincaré and Serre dualities, and it yields an isomorphism between Bott-Chern and Aeppli cohomologies.

This is an important issue, for example, in defining Massey products for Bott-Chern cohomology. In [AT15b], see also subsequent works by Tardini and Tomassini, triple Aeppli-Bott-Chern Massey products are defined, starting from Bott-Chern class, and yielding a class in Aeppli cohomology, up to indeterminacy.

If we want to force Massey products to be defined inside Bott-Chern cohomology, we would like to ask for the following property.

Say that a compact complex manifold X of complex dimension n satisfy the *Schweitzer qualitative property* if the natural pairing

$$H^{\bullet,\bullet}_{BC}(X) \times H^{\bullet,\bullet}_{BC}(X) \to \mathbb{C}\,, \qquad ([\alpha],[\beta]) \mapsto \int_X \alpha \wedge \beta$$

induced by the wedge product and by the pairing with the fundamental class $[X]$ is non-degenerate.

Unfortunately:

Theorem II.8.2 (Angella-Tardini [AT17])**.** *Let X be a compact complex manifold. If it satisfies the Schweitzer qualitative property, then it satisfies the $\partial\overline{\partial}$-Lemma.*

Proof. We prove that X satisfies the $\partial\overline{\partial}$-Lemma if and only if

$$\sum_{k\in\mathbb{Z}} \left| h^k_{BC} - h^k_A \right| = 0.$$

Recall that the Bott-Chern cohomology counts the corners with possible incoming arrows, and the Aeppli cohomology counts the corners with possible outcoming arrows, with the exceptions of squares. Therefore the hypothesis can be restated as: for any anti-diagonal, the number of ingoing arrows equals the number of outgoing arrows, except for squares. Since no ingoing arrow can enter the anti-diagonal of total degree 0, it follows that there is no zigzag of positive length in the whole diagram. That is, the $\partial\overline{\partial}$-Lemma holds. □

II.9. Generalized complex geometry and cohomological decompositions on symplectic manifolds

In this section, we frame complex geometry in the setting of generalized-complex geometry and then we draw the parallel between the cohomological theory of complex and symplectic manifolds.

Let X be a compact differentiable manifold of dimension $2n$. Note that a complex structure is given by an endomorphism $J\colon TX \to TX$ satisfying an algebraic condition ($J^2 = -1$)

and an analytic condition ($\mathrm{Nij}_J = 0$). Similarly, a symplectic structure is given by a non-degenerate 2-form that can be interpreted as an isomorphism $\omega \colon TX \to T^*X$ with an algebraic condition (being a 2-form) and an analytic condition ($d\omega = 0$).

Therefore, more in generale, let us consider the bundle $TX \oplus T^*X$ endowed with the natural symmetric pairing

$$\langle X + \xi | Y + \eta \rangle \;:=\; \frac{1}{2} \left(\xi(Y) + \eta(X) \right) \,.$$

Fix a d-closed 3-form H on X. On the space $\mathcal{C}^\infty(X; TX \oplus T^*X)$ of smooth sections of $TX \oplus T^*X$ over X, define the *H-twisted Courant bracket* as

$$[\_,\, \_]_H \colon \mathcal{C}^\infty(X; TX \oplus T^*X) \times \mathcal{C}^\infty(X; TX \oplus T^*X) \to \mathcal{C}^\infty(X; TX \oplus T^*X) \,,$$

$$[X + \xi,\, Y + \eta]_H \;:=\; [X,\, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d \left(\iota_X \eta - \iota_Y \xi \right) + \iota_Y \iota_X H$$

(where ι_X denotes the interior product with $X \in \mathcal{C}^\infty(X; TX)$ and $\mathcal{L}_X := [\iota_X,\, d]$ denotes the Lie derivative along $X \in \mathcal{C}^\infty(X; TX)$); the H-twisted Courant bracket can be seen also as a derived bracket induced by the H-twisted differential $d_H := d + H \wedge \_$, see [Gua04, §3.2], [Gua11, §2].

An *H-twisted generalized complex structure* on X, [Gua04, Definition 4.14, Definition 4.18], [Gua11, Definition 3.1] is an endomorphism $\mathcal{J} \in \mathrm{End}(TX \oplus T^*X)$ such that

(i) $\mathcal{J}^2 = -\mathrm{id}_{TX \oplus T^*X}$, and
(ii) $\mathcal{J}$ is orthogonal with respect to $\langle \_ | \_ \rangle$, and
(iii) the Nijenhuis tensor

$$\mathrm{Nij}_{\mathcal{J},H} \;:=\; -[\mathcal{J}\,\_,\, \mathcal{J}\,\_]_H + \mathcal{J}\,[\mathcal{J}\,\_,\, \_]_H + \mathcal{J}\,[\_,\, \mathcal{J}\,\_]_H + \mathcal{J}\,[\_,\, \_]_H$$

of $\mathcal{J}$ with respect to the H-twisted Courant bracket vanishes identically.

Equivalent definitions are given in [Gua04, Proposition 4.3] in terms of the $\sqrt{-1}$-eigenbundle of the $\mathbb{C}$-linear extension of $\mathcal{J}$ to $(TX \oplus T^*X) \otimes_{\mathbb{R}} \mathbb{C}$), and in [Gua04, Theorem 4.8] in terms of the Clifford action.

By definition, the *type* of a generalized complex structure $\mathcal{J}$ on X, [Gua04, §4.3], [Gua11, Definition 3.5], is the upper-semi-continuous function

$$\mathrm{type}(\mathcal{J}) \;:=\; \frac{1}{2} \dim_{\mathbb{R}} \left(T^*X \cap \mathcal{J} T^*X \right) .$$

A generalized complex structure $\mathcal{J}$ on X induces a $\mathbb{Z}$-graduation on the space of complex differential forms on X, see [Gua04, §4.4], [Gua11, Proposition 3.8], whose pieces will be denoted as $U^\bullet_{\mathcal{J}_J}$. Accordingly, the differential splits in the components $\partial_{\mathcal{J}_J}$ and $\overline{\partial}_{\mathcal{J}_J}$. One can then consider generalized Dolbeault and Bott-Chern cohomologies, and defne the $d_H d^{\mathcal{J}}_H$-Lemma property. An analogue of the Frölicher inequality for generalized Bott-Chern cohomology holds [AT15a].

Symplectic structures and complex structures provide the fundamental examples of generalized complex structures; in fact, the following generalized Darboux theorem by M. Gualtieri holds. (Recall that a *regular point* of a generalized complex manifold is a point at which the type of the generalized complex structure is locally constant.)

Theorem II.9.1 ([Gua04, Theorem 4.35], [Gua11, Theorem 3.6])**.** *For any regular point of a $2n$-dimensional generalized complex manifold with type equal to k, there is an open neighbourhood endowed with a set of local coordinates such that the generalized complex structure is a B-field transform of the standard generalized complex structure of $\mathbb{C}^k \times \mathbb{R}^{2n-2k}$.*

The standard generalized complex structure of constant type n (that is, locally equivalent to the standard complex structure of $\mathbb{C}^n$), the generalized complex structure of constant type 0 (that is, locally equivalent to the standard symplectic structure of $\mathbb{R}^{2n}$), and the B-field transform of a generalized complex structure are then the fundamental blocks.

- Let X be a compact $2n$-dimensional manifold endowed with a complex structure J [Gua04, Example 4.11, Example 4.25]. Consider the (0-twisted) generalized complex structure
$$\mathcal{J}_J := \left(\begin{array}{c|c} -J & 0 \\ \hline 0 & J^* \end{array}\right) ,$$
where J^* denotes the dual endomorphism of J. Hence, one gets the decomposition in types [Gua04, Example 4.25],
$$U^{\bullet}_{\mathcal{J}_J} = \bigoplus_{p-q=\bullet} \wedge^{p,q}_J X ,$$
and that
$$\partial_{\mathcal{J}_J} = \partial_J \qquad \text{and} \qquad \overline{\partial}_{\mathcal{J}_J} = \overline{\partial}_J .$$
The $d_H d^{\mathcal{J}}_H$-Lemma property correpondes then to the classical $\partial\overline{\partial}$-Lemma property.
- Let X be a compact $2n$-dimensional manifold endowed with a symplectic structure $\omega \in \wedge^2 X \simeq \mathrm{Hom}\,(TX; T^*X)$ [Gua04, Example 4.10]. Consider the (0-twisted) generalized complex structure
$$\mathcal{J}_\omega := \left(\begin{array}{c|c} 0 & -\omega^{-1} \\ \hline \omega & 0 \end{array}\right) ,$$
where $\omega^{-1} \in \mathrm{Hom}\,(T^*X; TX)$ denotes the inverse of $\omega \in \mathrm{Hom}\,(TX; T^*X)$. One gets the type decomposition [Cav06, Theorem 2.2],
$$U^{n-\bullet}_{\mathcal{J}_\omega} = \exp\left(\sqrt{-1}\omega\right) \left(\exp\left(\frac{\Lambda}{2\sqrt{-1}}\right) \left(\wedge^\bullet X \otimes \mathbb{C}\right)\right) ,$$
where $\Lambda := -\iota_{\omega^{-1}}$. By considering the natural isomorphism
$$\varphi \colon \wedge^\bullet X \otimes \mathbb{C} \to \wedge^\bullet X \otimes \mathbb{C} ,$$
$$\varphi(\alpha) := \exp\left(\sqrt{-1}\omega\right) \left(\exp\left(\frac{\Lambda}{2\sqrt{-1}}\right) \alpha\right) ,$$
one gets that, [Cav06, Corollary 1],
$$\varphi\left(\wedge^\bullet X \otimes \mathbb{C}\right) \simeq U^{n-\bullet} ,$$
$$\varphi d = \overline{\partial}_{\mathcal{J}_\omega} \varphi \qquad \text{and} \qquad \varphi d^{\mathcal{J}_\omega} = -2\sqrt{-1}\partial_{\mathcal{J}_\omega} \varphi .$$

In particular, one gets the symplectic Frölicher inequality [AT14]

$$\dim H^{\bullet}_{(d,d^{\Lambda};dd^{\Lambda})}(X) + \dim H^{\bullet}_{(dd^{\Lambda};d,d^{\Lambda})}(X) \;\geq\; 2\,\dim H^{\bullet}_{dR}(X;\mathbb{R}) \;,$$

and the equality holds if and only if X satisfies the Hard Lefschetz Condition:

Theorem II.9.2 ([Mat95, Corollary 2], [Yan96, Theorem 0.1], [Mer98, Proposition 1.4], [Gui01], [TY12, Proposition 3.13], [Cav05, Theorem 5.4], [AT14, Remark 2.3])**.** *Let X be a compact manifold endowed with a symplectic structure ω. The following conditions are equivalent:*

(1) every de Rham cohomology class of X admits a representative being both d-closed and d^{Λ}-closed, namely, Brylinski's conjecture [Bry88, Conjecture 2.2.7] *holds on X;*

(2) the Hard Lefschetz Condition holds on X;

(3) the natural map $H^{\bullet}_{(d,d^{\Lambda};dd^{\Lambda})}(\wedge^{\bullet}X) \to H^{\bullet}_{dR}(X;\mathbb{R})$ induced by the identity is surjective;

(4) the natural map $H^{\bullet}_{(d,d^{\Lambda};dd^{\Lambda})}(\wedge^{\bullet}X) \to H^{\bullet}_{dR}(X;\mathbb{R})$ induced by the identity is an isomorphism;

(5) the bounded $\mathbb{Z}$-graded $\mathbb{R}$-vector space $\wedge^{\bullet}X$ endowed with the endomorphisms $d \in \mathrm{End}^{1}(\wedge^{\bullet}X)$ and $d^{\Lambda} \in \mathrm{End}^{-1}(\wedge^{\bullet}X)$ satisfies the dd^{Λ}-Lemma;

(6) the decomposition

$$H^{\bullet}_{dR}(X;\mathbb{R}) \;=\; \bigoplus_{r\in\mathbb{N}} L^{r}\, H^{(0,\bullet-2r)}_{\omega}(X;\mathbb{R}) \;,$$

holds;

(7) the equality in $\dim H^{\bullet}_{(d,d^{\Lambda};dd^{\Lambda})}(X) + \dim H^{\bullet}_{(dd^{\Lambda};d,d^{\Lambda})}(X) \;\geq\; 2\,\dim H^{\bullet}_{dR}(X;\mathbb{R})$ holds for any degree.

Lecture III. Special Hermitian metrics on complex manifolds.

In this Lecture, we focus on special metrics on complex manifolds, in particular, we propose and study an analogue of the Yamabe problem for Hermitian manifolds.

References for this Lecture are [Huy05, Gau97, ACS17, ACS19]

III.10. Geometry of Hermitian and Kähler metrics on complex manifolds

A Hermitian metric h on a complex manifold X of complex dimension n is a smoothly varying Hermitian form (*i.e.* sesquilinear positive-definite form) on each fibre of the holomorphic tangent bundle $T^{1,0}X$. In this section, we first recall some basic notions of *Hermitian geometry*, here including the notion of *Chern connection* (more details can be found *e.g.* in [Bło13, Voi02a, Bal06, Mor07, Gau15, Huy05, Aub98, Bes08]; in particular, we try to follow closely [Bło13, §2] for notations), then we focus on the metric aspects of *Kähler geometry* (the reader can consult *e.g.* the references [Bło13, Voi02a, Bal06, Mor07, Huy05, Szé14, Joy07, War83, Ara12, Wel08, BDP02, Sch07]).

III.10.1. **Hermitian metrics on complex manifolds.** Let X be a complex manifold of complex dimension n, which is assumed to be connected; denote by J its complex structure.

A *Hermitian structure*

$$h = g - \sqrt{-1}\,\omega$$

on a complex manifold X of complex dimension n is a smoothly varying Hermitian form (*i.e.* sesquilinear positive-definite form) on each fibre of the holomorphic tangent bundle $T^{1,0}X$. In other words:

- $h := \langle \_ | \_ \rangle$ is a *Hermitian structure* on X:
  here, $h_x \colon T^{1,0}_x X \times T^{1,0}_x X \to \mathbb{C}$ is a sesquilinear positive-definite structure on $T^{1,0}_x X$, varying smoothly with $x \in X$; note that $(T_x X, J)$ can be identified with $(T^{1,0}_x X, \sqrt{-1})$ as $\mathbb{C}$-linear vector space, so h_x can be considered as a Hermitian structure on $T_x X$ as a $\mathbb{C}$-vector space with respect to J, as well as on $T_x X \otimes_{\mathbb{R}} \mathbb{C}$ with the $\mathbb{C}$-linear extension of J; in other words, we can set $h_x(\bar{X}, \bar{Y}) := h_x(X, Y)$ and extend h as a sesquilinear form on $T_x X \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}_x X \oplus \overline{T^{1,0}_x X}$ by linearity;
- $g := \Re h$:
  it is (the $\mathbb{C}$-linear extension, say complexification of) a *Riemannian structure* on the underlying smooth manifold, such that it is J-invariant, namely, $g(J\_, J\_) = g$;
- $\omega := -\Im h$:
  it is (the $\mathbb{C}$-linear extension, say complexification of) a real 2-form, such that it is J-invariant, namely, $\omega(J\_, J\_) = \omega$, and positive on the J-lines, namely, $\omega(X, JX) > 0$ for any non-zero X; it is called the *associated* $(1,1)$*-form*;
- they are related by:

$$\omega \;=\; g(J\_, \_)\,, \qquad g \;=\; \omega(\_, J\_)\,.$$

In the following, we will often confuse h, g, ω.

Note that any complex manifold is orientable. Indeed, fix any local frame $\{u^1, \ldots, u^n\}$ of TX as $\mathbb{C}$-vector bundle, and consider the orientation given by the local frame $\{u^1, Ju^1, \ldots, u^n, Ju^n\}$ as $\mathbb{R}$-vector bundle; the orientation does not depend on the choice of the initial local frame because of the Cauchy-Riemann equations. Since ω is almost-symplectic, it defines a canonical *volume form*:

$$\mathrm{vol} \;:=\; \omega^n,$$

(which is often replaced by $\frac{1}{n!}\omega^n$.)

The Hermitian structure is a pointwise-Hermitian structure on vector fields. It gives a duality

$$|\_\rangle \colon TX \to T^*X$$

between tangent and cotangent bundle. By means of this duality, it induces pointwise-Hermitian structure on 1-forms. By extending it to the exterior powers, we get a *pointwise-Hermitian structure* on forms: more precisely, it is given by the determinant on simple elements, extended to zero on forms of different degrees. We will denote it by $\langle \_ | \_ \rangle$.

If X is compact, we have a Hermitian L^2-*pairing* on global differential forms by integrating the pointwise-pairing with respect to the measure of the volume:

$$\langle\alpha|\beta\rangle_{\mathrm{L}^2} := \int_X \langle\alpha|\beta\rangle\, \omega^n .$$

The ($\mathbb{C}$-linear) *Hodge-$*$-operator* is defined by:

$$*\colon \wedge^{p,q} X \to \wedge^{n-q,n-p} X , \qquad \langle\alpha|\beta\rangle\, \omega^n = \alpha \wedge \overline{*\beta} .$$

It satisfies $*\lfloor^2_{\wedge^k X} = (-1)^{k(2n-k)}\,\mathrm{id} = (-1)^k\,\mathrm{id}$. In abstract term, it is given by the composition $p^{-1} \circ m$, where: $p\colon \wedge^{n-p,n-q} X \to \mathrm{Hom}_{\mathbb{C}}(\wedge^{p,q}X; \wedge^{n,n}X) \simeq \mathrm{Hom}_{\mathbb{C}}(\wedge^{p,q}X;\mathbb{C})$ is given by wedge product and by the $\mathbb{C}$-linear identification $\wedge^{n,n}X \simeq \mathbb{C}$ thanks to connectedness; and $m\colon \wedge^{p,q} X \to \mathrm{Hom}_{\mathbb{C}}(\wedge^{p,q}X;\mathbb{C})$ is given by the Hermitian structure.

III.10.2. **Levi-Civita connection.** Let us focus on the Riemannian structure g. Any tensor is extended by $\mathbb{C}$-linearity to the complex tangent bundle.

We recall that, on a Riemannian manifold, the *Levi-Civita connection*

$$\nabla := \nabla^{LC}\colon \mathcal{C}^\infty(X;TX) \to \mathcal{C}^\infty(X; TX \otimes T^*X)$$

is the unique connection (*i.e.* $\mathbb{R}$-linear map satisfying the Leibniz rule) on the tangent bundle being compatible with the metric and torsion-free, that is,

$$\nabla g = 0, \qquad T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] = 0.$$

where T is the *torsion.* It is given by:

$$\begin{aligned} 2\,g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad + g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y). \end{aligned}$$

In general, take a local coordinate system $\{x^h\}_h$. The (non-tensorial) *Christoffel symbols* determine the connection:

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^h} = \Gamma^k_{jh} \frac{\partial}{\partial x^k}.$$

The *Riemann curvature* is

$$\begin{aligned} R(X,Y) &:= \nabla^2_{X,Y} - \nabla^2_{Y,X} = (\nabla_X\nabla_Y - \nabla_{\nabla_X Y}) - (\nabla_Y\nabla_X - \nabla_{\nabla_Y X}) \\ &= [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \in \wedge^2(X; \mathrm{End}(TX)). \end{aligned}$$

We also use the tensor:

$$R(X,Y,Z,W) = g(R(X,Y)Z,W).$$

The Levi-Civita connection satisfies the further symmetry known as *first (algebraic) Bianchi identity*:

$$\sum_{\sigma\in\mathfrak{S}} R(\sigma x, \sigma y)\sigma z = 0. \tag{1Bnc}$$

(This follows by the Levi-Civita connection being torsion-free, and by the Jacobi identity for the Lie bracket $[\_, \_]$.) More in general, for a metric connection ∇ with possibly non-zero torsion T, we have (compare *e.g.* [Gau15, §1.16]):

$$\sum_{\sigma \in \mathfrak{S}} R(\sigma x, \sigma y)\sigma z = d^{\nabla} T(x, y, z).$$

In particular, it follows $g(R(X,Y)Z, W) = -g(R(X,Y)W, Z)$, and

$$R^{\nabla} \in S^2 \wedge^2 X. \tag{Symm}$$

We also have the *second (differential) Bianchi identity*:

$$(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0$$

(it can be read as $d^{\nabla} R = 0$ [Gau15, page 44]).

The *first Ricci curvature* and the *second Ricci curvature* are, respectively, the traces

$$\mathrm{Ric}^{(i)}(x,y) = \mathrm{tr}\, R(x,y) \in \wedge^2 X,$$

$$\mathrm{Ric}^{(ii)}(z,w) = \mathrm{tr}_g R(\_, \_, z, w) \in \wedge^2 X \subset \mathrm{End}(TX).$$

Thanks to (Symm), they coincide. The *scalar curvature* is

$$\mathrm{Scal} = \mathrm{tr}_g \mathrm{Ric}^{(i)} = \mathrm{tr}\, \mathrm{Ric}^{(ii)} \in \mathcal{C}^{\infty}(X; \mathbb{R}).$$

The *sectional curvature* at a point p of a plane $\sigma_p := \mathrm{span}\{X, Y\}$ in the tangent space at p is the Gaussian curvature of the surface which has the plane σ_p as a tangent plane at p. More precisely,

$$\mathrm{K}(X,Y) := \frac{g(R(X,Y)X,Y)}{|X|^2|Y|^2 - g(X,Y)}.$$

The sectional curvature determines the curvature tensor completely.

III.10.3. **Chern connection of Hermitian metrics.** We have seen that, in general, the Levi-Civita connection does not preserve the complex structure. This happens only for Kähler metrics. Hence we consider *Hermitian connections*, namely, connections on the holomorphic tangent bundle preserving the Hermitian structure: that is, both metric and complex structure. In general they have torsion. Prescribed torsion components $T_b^{1,1} = 0$ yields to the notion of *canonical connection* in the Lichnerowicz-Gauduchon family [Gau97].

In particular, we focus now on the *Chern connection* ∇^{Ch}. It is the unique connection on $T^{1,0}X$ preserving g and J, (*i.e.* $\nabla^{Ch} g = \nabla^{Ch} J = 0$,) and such that its part of type $(0,1)$ coincides with the Cauchy-Riemann operator $\overline{\partial}$ associated to the holomorphic structure, (equivalently, its curvature is a $(1,1)$-form,) [Gau97]. We will see that, (up to $\mathbb{C}$-linear extension,) it coincides with the Levi-Civita connection in case of Kähler metrics. The same definition work to define the Chern connection of a holomorphic vector bundle $E \to X$, reducing to the former in case $E = T^{1,0}X$.

The Chern connection $\nabla := \nabla^{Ch}$ satisfies the *type condition* symmetry:

$$R^{\nabla} \in \wedge^{1,1}(X; \mathrm{End}(T^{1,0}X)),$$

namely,

$$R^\nabla(x,y,z,w) \;=\; R^\nabla(x,y,Jz,Jw) \;=\; R^\nabla(Jx,Jy,z,w)\;. \tag{Cplx}$$

(The J-invariance in the third and fourth arguments, namely the property $R^\nabla \in \wedge^2(X;\mathrm{End}(T^{1,0}X))$, follows from $\nabla J = 0$ and $g(J\_, J\_) = g(\_,\_)$. The conclusion follows from $\nabla^{0,1} = \overline{\partial}$ yielding $(\nabla^{0,1})^2 = 0$ and by ∇ being real.)

For Chern connection, some formulas take a more natural expression. *E.g.* the *Chern Ricci curvature* is locally given by

$$\mathrm{Ric}^{Ch} \stackrel{\mathrm{loc}}{=} \sqrt{-1}\overline{\partial}\partial\log\omega^n, \tag{5}$$

the *Chern scalar curvature* is

$$\mathrm{S}^{Ch} = \sqrt{-1}\,\mathrm{tr}_\omega\overline{\partial}\partial\log\omega^n,$$

and the *Chern Laplacian* is

$$\Delta^{Ch}\varphi = \langle\omega|dd^c\varphi\rangle = 2\sqrt{-1}\,\mathrm{tr}_\omega\overline{\partial}\partial\varphi.$$

This motivates the study of Kähler metrics: where we use compatibility to relate complex structure (Chern connection) and Riemannian properties (Levi-Civita connection). In this case, we derive further symmetries for the curvature tensors (compare [?]).

Remark III.10.1. Many of the results that follow make sense also for other canonical connections in the Lichnerowicz-Gauduchon family, that are characterized by

$$g(\nabla^t_x y, z) = g(\nabla^{LC}_x y, z) + \frac{1-t}{4}\,T(x,y,z) + \frac{1+t}{4}\,C(x,y,z),$$

where

$$T := Jd\omega := -d\omega(J\_, J\_, J\_), \qquad C := d\omega(J\_,\_,\_),$$

varying $t \in \mathbb{R}$. Distinguished connections are the Chern connection $\nabla^{Ch} = \nabla^1$ and the *Bismut connection* $\nabla^+ = \nabla^{-1}$.

III.11. Constant Chern-scalar curvature metrics in conformal classes

In Riemannian geometry, the *Yamabe problem*, answered by Yamabe, Trudinger, Aubin, and Schoen, states the existence of constant scalar curvature metrics in any conformal class. We state an analogue in the Hermitian context, making advantage of the Chern connection, and attempting to notions of canonical Hermitian metrics on complex manifolds

Let X be a compact complex manifold of complex dimension n. Once fixed ω Hermitian metric on X, we will denote by $\{\omega\}$ its conformal class. Note that any conformal metric is still Hermitian.

We look for $\{\omega' \in \{\omega\} \;:\; \mathrm{Scal}(\omega') \text{ is constant}\}$. Note that $\mathcal{G}_X(\{\omega\}) := \mathcal{HC}onf(X;\{\omega\}) \times \mathbb{R}^+$ acts on this space, by biholomorphisms preserving the conformal class and by homotheties respectively. We quotient out its action and we consider the *Chern-Yamabe modui space*

$$\mathcal{C}h\mathcal{Y}a(X;\{\omega\}) := \left\{\omega' \in \{\omega\} \;:\; \mathrm{Scal}(\omega') \text{ is constant}\right\}/\mathcal{G}_X(\{\omega\}).$$

We refer to the problem of studying $\mathcal{CY}(X;\{\omega\})$ (being empty, compact, ...) as the *Chern-Yamabe problem.*

Remark III.11.1. In the non-Kähler context, the Chern-Yamabe problem is actually different from the classical Yamabe problem, and from the almost-complex Yamabe problem introduced by Del Rio and Simanca in [dRS03]. This follows as Liu-Yang [LY12] showed that the totale Chern-scalar curvature being equal to the total scalar curvature (respectively, to the total J-scalar curvature) forces the metric to be Kähler.

It is not difficult to write down the transformation equation for the Chern-scalar curvature under conformal changes:

$$S^{Ch}(\exp\frac{2f}{n}\omega) = \exp(-\frac{2f}{n})\left(S^{Ch}(\omega)+\Delta^{Ch}_{\omega}(f)\right).$$

Here, on smooth functions,

$$\Delta^{Ch}_{\omega}(f) = \langle\omega|dd^cf\rangle_{\omega} = 2\sqrt{-1}\mathrm{tr}_{\omega}dd^cf$$

is the Chern-Laplacian, and it differs by the Hodge-de Rham Laplacian $\Delta_{d,\omega}=[d,d^*_{\omega}]$ by a first-order term:

$$\Delta^{Ch}(f)_{\omega} = \Delta_{d,\omega}+\langle df|\theta\rangle_{\omega}$$

where θ is called the (balanced) *Lee form* and is characterized by

$$d\omega^{n-1} = \theta\wedge\omega^{n-1}.$$

Two special cases are of interest:

- when $\theta=0$, equivalently, $d\omega^{n-1}=0$, namely, ω is known as *balanced* in the sense of Michelsohn [Mic82], then $\Delta^{Ch}=\Delta_d$ on functions;
- when $d^*_{\omega}\theta=0$, equivalently, $\partial\overline{\partial}\omega^{n-1}=0$ namely, ω is known as *standard* in the sense of Gauduchon [Gau77], then $\int_X\Delta^{Ch}f\omega^n=0$. A foundational theorem by Gauduhon assures the existence of a unique Gauduchon metric of volume one in any conformal class, that we will denote as η and take as reference metric.

Since we are mod out the homothethies, we can choose to normalize the conformal factor f in $\exp\frac{2f}{n}\omega\in\{\omega\}$ in order to have

$$\{\omega\}_1 := \left\{\exp\frac{2f}{n}\omega \ : \ \int_X\exp\frac{2f}{n}\frac{\eta^n}{n!}=1\right\}.$$

Note that this is *not* the volume one normalization.

At the end of the day, we are reduced to solve the equation

$$\Delta^{Ch}_{\eta}f+S^{Ch}(\eta) = \lambda\cdot\exp\frac{2f}{n} \tag{ChYa}$$

for f smooth function and λ constant, equal to the expected constant Chern-scalar curvature. Thanks to the chosen normalization,

$$\lambda = \int_X S^{Ch}(\eta)\frac{\eta^n}{n!} = \frac{1}{(n-1)!}\int_X c_1^{BC}(K_X^{-1})\wedge[\eta^{n-1}]$$

is an invariant depending just on the complex structure and on the conformal class, and it is known as *Gauduchon degree* of the conformal class $\Gamma_X(\{\omega\})$.

Here $\Gamma_X(\{\omega\})$ is the degree of the anticanonical line bundle K_X^{-1} with respect to the Gaduchon metric in $\{\omega\}$. Then, by [Gau77], it is equal to the volume of the divisor associated to any meromorphic section of K_X^{-1} by means of η. In particular:

Proposition III.11.2 ([Gau77]). *If* $\mathrm{Kod}(X) \geq 0$*, then any conformal class has* $\Gamma_X(\{\omega\}) \leq 0$.

On the other hand, by Teleman, Inoue surfaces have $\mathrm{Kod}(X) = -\infty$ and $\Gamma_X(\{\omega\}) \leq 0$ for any conformal class.

Theorem III.11.3 (DA-Calamai-Spotti [ACS17]). *If* $\Gamma_X\{\omega\} \leq 0$*, then* $\mathcal{CY}(X; \{\omega\}) = \{\omega_p\}$.

That is, for conformal classes with non-positive Gauduchon degree, we have existence and uniqueness; moreover, such a unique metric is canonical in the sense that $\mathcal{HC}onf(X; \{\omega\}) = \mathcal{HI}som(X; \omega_p)$.

Proof. The case $\Gamma = 0$ is reduced to a linear PDE

$$\Delta^{Ch} f = -S^{Ch}(\eta)$$

so it suffices to note that $-S^{Ch}(\eta) \in \mathbb{C}^\perp = (\ker(\Delta^{Ch})^*)^\perp = \mathrm{im}\Delta^{Ch}$ because of $\Gamma(\{\omega\} = \int S^{Ch}(eta)\frac{\eta^n}{n!} = 0$.

For the case $\Gamma < 0$, we first use conformal technique to find a reference metric in the conformal class such that $S^{Ch}(\omega) < 0$ everywhere. Then we apply the continuity method to

$$\Delta^{Ch}(f) + tS^{Ch}(\omega) = \lambda \exp\frac{2f}{n} - \lambda(1-t).$$

The openness follows by the linearization $Dv = \Delta^{Ch}v - \lambda\exp\frac{2f}{n}\frac{2v}{n}$ being elliptic, thanks to $\mathrm{index}D = \mathrm{index}\Delta^{Ch} = 0$.

The closedness follows by uniform L^∞-bounds thanks to the Maximum Principle, where the sign of the Gauduchon degree play a role.

The standard theory by Calderon-Zygmund, Ascoli-Arzelà, bootstrap, Schauder conclude the existence. The uniqueness follows by Maximum Principle. □

Some remarks in the positive case:

- Hopf surfaces $\mathbb{C}^2 \setminus 0/(z \mapsto 2z)$ have $\mathrm{Kod} = -\infty$ and $\Gamma(\{\omega\}) \geq 0$ for any conformal class. The standard metric $\omega_{\mathbb{C}^2}/|z|^2$ has constant positive Chern-scalar curvature.
- Implicit function theorem applies when $\|S^{Ch}(\omega)\|_{\mathcal{C}^{0,\alpha}}$ small, which gives more examples.
- Assuming bounds on the volumes, we can prove compactness of the Chern-Yamabe moduli space even if non-empty.

- As de Lima-Piccione-Zedda [dLPZ12] do in the classical Yamabe case, we can use (non-variational) bifurcation techniques to show non-uniqueness in the positive curvature case.
- The Chern-Yamabe problem is in general non-variational. Indeed, it is variational if and only if $h \mapsto \int_X h\langle df|\theta\rangle_\omega \frac{\omega^n}{n!}$ is cloased (whence exact), but this happens if and only if $\theta = 0$, that is, the conformal class is balanced. Even in this case, we are not able to bound the functional $\mathcal{F}(f) = \frac{1}{2}\int_X |df|^2_\eta \frac{\eta^n}{n!} + \int_X S^{Ch}(\eta)\frac{\eta^n}{n!}$.

III.12. Chern-Einstein metrics

As we have seen, there are *a priori* three different ways to trace the Chern curvature

$$\Theta(\omega) \stackrel{\text{loc}}{=} \Theta_{i\bar{j}k\bar{\ell}}\sqrt{-1}dz^j \wedge d\bar{z}^j \otimes \sqrt{-1}dz^k \wedge d\bar{z}^\ell.$$

- $\mathrm{Ric}^1 := \mathrm{tr}\Theta(\omega) \stackrel{\text{loc}}{=} h^{\bar{\ell}k}\Theta_{i\bar{j}k\bar{\ell}}\sqrt{-1}dz^i \wedge d\bar{z}^j \in c_1^{BC}(X) \in H^{1,1}_{BC}(X)$;
- $\mathrm{Ric}^2 \stackrel{\text{loc}}{:=} h^{\bar{j}i}\Theta_{i\bar{j}k\bar{\ell}}\sqrt{-1}dz^k \wedge d\bar{z}^\ell \in \wedge^{1,1}_{\mathbb{R}}X$;
- $\mathrm{Ric}^3 :=\stackrel{\text{loc}}{:=} h^{\bar{\ell}i}\Theta_{i\bar{j}k\bar{\ell}}d\bar{z}^j \otimes dz^k$.

We have then three different *Chern-Einstein problems*, $j \in \{1,2,3\}$:

$$\mathrm{Ric}^j(\omega) = \lambda\omega$$

for some smooth Einstein factor λ (non necessarily constant).

As for the first Chern-Einstein problem, we notice that, either $\lambda = 0$ and then the metric is *non-Kähler Calabi-Yau* in the sense of Tosatti [Tos15]; or $\lambda \neq 0$ and the metric is conformally-Kähler [ACS19]. Therefore, in the non-Kähler situations, the only interesting case is completely covered by the following:

Theorem III.12.1 (Tosatti [Tos15]). *If $c_1^{BC}(X) = 0$, then any conformal class contains a Chern-Ricci-flat metric.*

As for the second-Chern-Einstein problem, we note that:

- A second-Ch-Ei metric is Hermitian-Einstein on the holomorphic tangent bundle with respect to itself. It follows that Bogomolov-Lübke and Kobayashi-Hitchin give obstructions.
- if the Ch-Ya conjecture on the existence of constant Chern-scalar curvature metrics in any conformal class holds, then we can assume λ to be constant.
- We have some examples. In case $\lambda > 0$ just take Hopf surfaces, or the homogeneous spaces by Podestà [Pod18]. In case $\lambda = 0$, we have non-compact solvable Lie groups. In case $\lambda < 0$, we have non-compact Lie groups; see [ACS19]. We do not know any compact example in the non-positive curvature case.

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$H^k_{dR}(X;\mathbb{C})$	g-harmonic representatives	$\dim_{\mathbb{C}} H^k_{dR}(X;\mathbb{C})$
$k=1$	$dz_1,\ d\bar{z}_1$	2
$k=2$	$dz_{23},\ dz_{1\bar{1}},\ dz_{2\bar{3}},\ dz_{3\bar{2}},\ dz_{\bar{2}\bar{3}}$	5
$k=3$	$dz_{123},\ dz_{23\bar{1}},\ dz_{12\bar{3}},\ dz_{13\bar{2}},\ dz_{1\bar{2}\bar{3}},\ dz_{2\bar{1}\bar{3}},\ dz_{3\bar{1}\bar{2}},\ dz_{\bar{1}\bar{2}\bar{3}}$	8
$k=4$	$dz_{123\bar{1}},\ dz_{12\bar{1}\bar{3}},\ dz_{23\bar{2}\bar{3}},\ dz_{13\bar{1}\bar{2}},\ dz_{1\bar{1}\bar{2}\bar{3}}$	5
$k=5$	$dz_{123\bar{2}\bar{3}},\ dz_{23\bar{1}\bar{2}\bar{3}}$	2

TABLE 1. A basis of harmonic representatives for the completely-solvable Nakamura manifold with respect to the metric $g := dz_1 \odot d\bar{z}_1 + \exp^{-z_1-\bar{z}_1} dz_2 \odot d\bar{z}_2 + \exp^{z_1+\bar{z}_1} dz_3 \odot d\bar{z}_3$.

$H^{\bullet,\bullet}_{BC}(\Gamma\backslash G)$	**case *(i)***	**case *(ii)***	**case *(iii)***
$(\mathbf{0},\mathbf{0})$	$\mathbb{C}\langle 1\rangle$	$\mathbb{C}\langle 1\rangle$	$\mathbb{C}\langle 1\rangle$
$(\mathbf{1},\mathbf{0})$	$\mathbb{C}\langle [dz_1]\rangle$	$\mathbb{C}\langle [dz_1]\rangle$	$\mathbb{C}\langle dz_1\rangle$
$(\mathbf{0},\mathbf{1})$	$\mathbb{C}\langle [dz_{\bar1}]\rangle$	$\mathbb{C}\langle [dz_{\bar1}]\rangle$	$\mathbb{C}\langle dz_{\bar1}\rangle$
$(\mathbf{2},\mathbf{0})$	$\mathbb{C}\langle [\exp^{-z_1} dz_{12}],\ [\exp^{z_1} dz_{13}],\ [dz_{23}]\rangle$	$\mathbb{C}\langle [dz_{23}]\rangle$	$\mathbb{C}\langle dz_{23}\rangle$
$(\mathbf{1},\mathbf{1})$	$\mathbb{C}\langle [dz_{1\bar1}],\ [\exp^{-z_1} dz_{1\bar2}],\ [\exp^{z_1} dz_{1\bar3}],\ [dz_{2\bar3}],$ $[dz_{3\bar2}],\ [\exp^{-\bar z_1} dz_{2\bar1}],\ [\exp^{\bar z_1} dz_{3\bar1}]\rangle$	$\mathbb{C}\langle [dz_{1\bar1}],\ [dz_{2\bar3}],\ [dz_{3\bar2}]\rangle$	$\mathbb{C}\langle dz_{1\bar1},\ dz_{2\bar3},\ dz_{3\bar2}\rangle$
$(\mathbf{0},\mathbf{2})$	$\mathbb{C}\langle [dz_{\bar2\bar3}],\ [\exp^{-\bar z_1} dz_{\bar1\bar2}],\ [\exp^{\bar z_1} dz_{\bar1\bar3}]\rangle$	$\mathbb{C}\langle [dz_{\bar2\bar3}]\rangle$	$\mathbb{C}\langle dz_{\bar2\bar3}\rangle$
$(\mathbf{3},\mathbf{0})$	$\mathbb{C}\langle [dz_{123}]\rangle$	$\mathbb{C}\langle [dz_{123}]\rangle$	$\mathbb{C}\langle dz_{123}\rangle$
$(\mathbf{2},\mathbf{1})$	$\mathbb{C}\big\langle [\exp^{-z_1} dz_{12\bar1}],\ [\exp^{-2z_1} dz_{12\bar2}],\ [dz_{12\bar3}],\ [\exp^{z_1} dz_{13\bar1}],\ [dz_{13\bar2}],$ $[\exp^{2z_1} dz_{13\bar3}],\ [dz_{23\bar1}],\ [\exp^{-\bar z_1} dz_{12\bar1}],\ [\exp^{\bar z_1} dz_{13\bar1}]\big\rangle$	$\mathbb{C}\big\langle [dz_{23\bar1}],\ [\exp^{-2z_1} dz_{12\bar2}],\ [\exp^{2z_1} dz_{13\bar3}],$ $[dz_{12\bar3}],\ [dz_{13\bar2}]\big\rangle$	$\mathbb{C}\langle dz_{23\bar1},\ dz_{12\bar3},\ dz_{13\bar2}\rangle$
$(\mathbf{1},\mathbf{2})$	$\mathbb{C}\big\langle [\exp^{-\bar z_1} dz_{1\bar1\bar2}],\ [\exp^{-2\bar z_1} dz_{2\bar1\bar2}],\ [dz_{3\bar1\bar2}],\ [\exp^{\bar z_1} dz_{1\bar1\bar3}],\ [dz_{2\bar1\bar3}],$ $[\exp^{2\bar z_1} dz_{3\bar1\bar3}],\ [dz_{1\bar2\bar3}],\ [\exp^{-z_1} dz_{1\bar1\bar2}],\ [\exp^{z_1} dz_{1\bar1\bar3}]\big\rangle$	$\mathbb{C}\big\langle [dz_{1\bar2\bar3}],\ [\exp^{-2\bar z_1} dz_{2\bar1\bar2}],\ [\exp^{2\bar z_1} dz_{3\bar1\bar3}],$ $[dz_{2\bar1\bar3}],\ [dz_{3\bar1\bar2}]\big\rangle$	$\mathbb{C}\langle dz_{1\bar2\bar3},\ dz_{2\bar1\bar3},\ dz_{3\bar1\bar2}\rangle$
$(\mathbf{0},\mathbf{3})$	$\mathbb{C}\langle [dz_{\bar1\bar2\bar3}]\rangle$	$\mathbb{C}\langle [dz_{\bar1\bar2\bar3}]\rangle$	$\mathbb{C}\langle dz_{\bar1\bar2\bar3}\rangle$
$(\mathbf{3},\mathbf{1})$	$\mathbb{C}\langle [dz_{123\bar1}],\ [\exp^{-z_1} dz_{123\bar2}],\ [\exp^{z_1} dz_{123\bar3}]\rangle$	$\mathbb{C}\langle [dz_{123\bar1}]\rangle$	$\mathbb{C}\langle dz_{123\bar1}\rangle$
$(\mathbf{2},\mathbf{2})$	$\mathbb{C}\big\langle [\exp^{-2z_1} dz_{12\bar1\bar2}],\ [dz_{12\bar1\bar3}],\ [\exp^{-z_1} dz_{12\bar2\bar3}],\ [dz_{13\bar1\bar2}],$ $[\exp^{2z_1} dz_{13\bar1\bar3}],\ [\exp^{z_1} dz_{13\bar2\bar3}],\ [dz_{23\bar2\bar3}],\ [\exp^{-2\bar z_1} dz_{12\bar1\bar2}],$ $[\exp^{-\bar z_1} dz_{23\bar1\bar2}],\ [\exp^{2\bar z_1} dz_{13\bar1\bar3}],\ [\exp^{\bar z_1} dz_{23\bar1\bar3}]\big\rangle$	$\mathbb{C}\big\langle [dz_{12\bar1\bar3}],\ [\exp^{-2z_1} dz_{12\bar1\bar2}],\ [\exp^{-2\bar z_1} dz_{12\bar1\bar2}],$ $[\exp^{2z_1} dz_{13\bar1\bar3}],\ [\exp^{2\bar z_1} dz_{13\bar1\bar3}],\ [dz_{23\bar2\bar3}],\ [dz_{13\bar1\bar2}]\big\rangle$	$\mathbb{C}\langle dz_{12\bar1\bar3},\ dz_{23\bar2\bar3},\ dz_{13\bar1\bar2}\rangle$
$(\mathbf{1},\mathbf{3})$	$\mathbb{C}\langle [dz_{1\bar1\bar2\bar3}],\ [\exp^{-\bar z_1} dz_{2\bar1\bar2\bar3}],\ [\exp^{\bar z_1} dz_{3\bar1\bar2\bar3}]\rangle$	$\mathbb{C}\langle [dz_{1\bar1\bar2\bar3}]\rangle$	$\mathbb{C}\langle dz_{1\bar1\bar2\bar3}\rangle$
$(\mathbf{3},\mathbf{2})$	$\mathbb{C}\langle [\exp^{-z_1} dz_{123\bar1\bar2}],\ [\exp^{z_1} dz_{123\bar1\bar3}],\ [dz_{123\bar2\bar3}],$ $[\exp^{-\bar z_1} dz_{123\bar1\bar2}],\ [\exp^{\bar z_1} dz_{123\bar1\bar3}]\rangle$	$\mathbb{C}\langle [dz_{123\bar2\bar3}]\rangle$	$\mathbb{C}\langle dz_{123\bar2\bar3}\rangle$
$(\mathbf{2},\mathbf{3})$	$\mathbb{C}\langle [\exp^{-z_1} dz_{12\bar1\bar2\bar3}],\ [\exp^{z_1} dz_{13\bar1\bar2\bar3}],\ [dz_{23\bar1\bar2\bar3}],$ $[\exp^{-\bar z_1} dz_{12\bar1\bar2\bar3}],\ [\exp^{\bar z_1} dz_{13\bar1\bar2\bar3}]\rangle$	$\mathbb{C}\langle [dz_{23\bar1\bar2\bar3}]\rangle$	$\mathbb{C}\langle dz_{23\bar1\bar2\bar3}\rangle$
$(\mathbf{3},\mathbf{3})$	$\mathbb{C}\langle [dz_{123\bar1\bar2\bar3}]\rangle$	$\mathbb{C}\langle [dz_{123\bar1\bar2\bar3}]\rangle$	$\mathbb{C}\langle dz_{123\bar1\bar2\bar3}\rangle$

TABLE 2. The Bott-Chern cohomology of the completely-solvable Nakamura manifold, [AK17a, Example 2.17, Table 4, Table 5, Table 3].

$H^{\bullet,\bullet}_{\sharp}(X)$	dR	case *(i)* $\overline{\partial}$	case *(i)* BC	case *(ii)* $\overline{\partial}$	case *(ii)* BC	case *(iii)* $\overline{\partial}$	case *(iii)* BC
$(\mathbf{0},\mathbf{0})$	1	1	1	1	1	1	1
$(\mathbf{1},\mathbf{0})$	2	3	1	1	1	1	1
$(\mathbf{0},\mathbf{1})$		3	1	1	1	1	1
$(\mathbf{2},\mathbf{0})$	5	3	3	1	1	1	1
$(\mathbf{1},\mathbf{1})$		9	7	5	3	3	3
$(\mathbf{0},\mathbf{2})$		3	3	1	1	1	1
$(\mathbf{3},\mathbf{0})$	8	1	1	1	1	1	1
$(\mathbf{2},\mathbf{1})$		9	9	5	5	3	3
$(\mathbf{1},\mathbf{2})$		9	9	5	5	3	3
$(\mathbf{0},\mathbf{3})$		1	1	1	1	1	1
$(\mathbf{3},\mathbf{1})$	5	3	3	1	1	1	1
$(\mathbf{2},\mathbf{2})$		9	11	5	7	3	3
$(\mathbf{1},\mathbf{3})$		3	3	1	1	1	1
$(\mathbf{3},\mathbf{2})$	2	3	5	1	1	1	1
$(\mathbf{2},\mathbf{3})$		3	5	1	1	1	1
$(\mathbf{3},\mathbf{3})$	1	1	1	1	1	1	1

TABLE 3. The dimensions of the de Rham, Dolbeault, and Bott-Chern cohomologies of the completely-solvable Nakamura manifold, [Kas13b, Example 1], [AK17a, Example 2.17], see [AK17a, Table 6].

classes	$\mathbf{h}^{\mathbf{1}}_{\overline{\partial}}$	$\mathbf{h}^{\mathbf{1}}_{\mathbf{BC}}$	$\mathbf{h}^{\mathbf{1}}_{\mathbf{A}}$	$\mathbf{h}^{\mathbf{2}}_{\overline{\partial}}$	$\mathbf{h}^{\mathbf{2}}_{\mathbf{BC}}$	$\mathbf{h}^{\mathbf{2}}_{\mathbf{A}}$	$\mathbf{h}^{\mathbf{3}}_{\overline{\partial}}$	$\mathbf{h}^{\mathbf{3}}_{\mathbf{BC}}$	$\mathbf{h}^{\mathbf{3}}_{\mathbf{A}}$	$\mathbf{h}^{\mathbf{4}}_{\overline{\partial}}$	$\mathbf{h}^{\mathbf{4}}_{\mathbf{BC}}$	$\mathbf{h}^{\mathbf{4}}_{\mathbf{A}}$	$\mathbf{h}^{\mathbf{5}}_{\overline{\partial}}$	$\mathbf{h}^{\mathbf{5}}_{\mathbf{BC}}$	$\mathbf{h}^{\mathbf{5}}_{\mathbf{A}}$
(i)	5	4	6	11	10	12	14	14	14	11	12	10	5	6	4
(ii.a)	4	4	6	9	8	11	12	14	14	9	11	8	4	6	4
(ii.b)	4	4	6	9	8	10	12	14	14	9	10	8	4	6	4
(iii.a)	4	4	6	8	6	11	10	14	14	8	11	6	4	6	4
(iii.b)	4	4	6	8	6	10	10	14	14	8	10	6	4	6	4
	$\mathbf{b_1 = 4}$			$\mathbf{b_2 = 8}$			$\mathbf{b_3 = 10}$			$\mathbf{b_4 = 8}$			$\mathbf{b_5 = 4}$		

TABLE 4. Dimensions of cohomologies for the small deformations of the Iwasawa manifold.